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A Poincaré Lemma for Connection Forms

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Denote by $\mathscr{P}$ the space of piecewise smooth curves in R^n beginning at the origin. A path 2-form is a function h on $\mathscr{P}$ such that for each element σ in $\mathscr{P}$, $h(\sigma)$ is a 2-form at the endpoint of σ with values in a Lie algebra $\mathscr{G}$. For example, if A is a smooth $\mathscr{G}$ valued connection form on R^n with curvature F and parallel translation operator $P(\sigma)$ then the equation $L^A(\sigma) = P(\sigma)^{-1} F(\sigma(1)) P(\sigma)$ defines L^A as a path 2-form. A necessary and sufficient condition is given to characterize those path 2-forms which arise in this way. By way of application it is shown that the Birula–Mandelstam generalization of Maxwell's equations to nonabelian gauge fields is equivalent to the Yang–Mills equation.

1. INTRODUCTION

It is known, at a partly informal level, that the construction of quantized Yang–Mills fields in n space–time dimensions can be accomplished by giving suitable meaning as a probability measure, to the expression

$$d\mu(A) = Z^{-1} \exp\left[\int_{R^n} \sum_{i<j} \operatorname{trace}(F_{ij}(x)^2)\, dx\right] \mathscr{D}A, \tag{1.1}$$

wherein A runs over some space $\mathscr{A}$ of connection forms on R^n, $F_{ij}(x) := \partial_i A_j - \partial_j A_i + [A_i(x), A_j(x)]$ is the curvature form of A, $\mathscr{D}A := \prod_{i=1}^n \prod_{x \in R^n} d(A_i(x))$ is an infinite dimensional "Lebesgue measure" on $\mathscr{A}$, and Z is a normalization constant [1]. A takes its values in the Lie algebra $\mathscr{G}$ of some subgroup G of the unitary group $U(N)$.

If G is a commutative group, e.g., the circle group $U(1)$, it is well understood that the quadratic form in the exponent of (1.1) has an infinite dimensional kernel and that the "measure" μ therefore appears as a meaningful infinite dimensional Gaussian measure times a meaningless infinite dimensional Lebesgue measure. The customary interpretation of (1.1) in this case amounts, in effect, to ignoring the Lebesgue measure fac-

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tor and interpreting μ as the Gaussian measure factor on the quotient space obtained by dividing out the space $\mathcal{A}$ by the kernel of the quadratic form. Minlos' theorem ensures that this procedure will indeed yield a probability measure μ on the quotient space provided one starts with a very large space $\mathcal{A}$, say $\mathcal{A} = \{A: A_i \in \mathcal{S}'(R^n, \mathcal{G})\}$. It will fail, however, if one starts with too small a space such as $\mathcal{A}^\infty := \{A: A_i \in C^\infty(R^n; \mathcal{G})\}$ or $\{A: \text{exponent in (1.1) is finite}\}$ or even $\{A: A_i \in L^1_{\text{loc}}(R^n; \mathcal{G})\}$ (cf. [13, 14, 24, 38, 40]).

If G is not commutative then the curvature F is quadratic in A and the exponent in (1.1) is quartic in A. The resulting object μ is in no sense Gaussian and is therefore harder to give a precise meaning to as a measure. To give meaning to μ as a probability measure is an entirely open problem in space–time dimensions $n \geqslant 3$. The kernel problem, described above in the commutative case, persists in the noncommutative case in the following form. If $g \in C^\infty(R^n; G)$ then the map $A \to A^g$ defined by

$$A^g(x) = g(x)^{-1} A(x)\, g(x) + g(x)^{-1}\, dg(x) \tag{1.2}$$

leaves the exponent in (1.1) invariant while "translating" the infinite dimensional "Lesbesgue measure" $\mathcal{D}A$. (If $G = U(1)$ then $g^{-1}\, dg$ is in the kernel since its curvature, $d(g^{-1}\, dg)$, is zero.) As a result, one must, as in the commutative case, seek to interpret μ as a measure in a space $\mathcal{M}$ which is a completion, in some kind of generalized function sense, of the so-called gauge orbit space $\mathcal{M}^\infty := \mathcal{A}^\infty / C^\infty(R^n; G)$. The geometry of $\mathcal{M}^\infty$ has been discussed in [5, 6, 32, 44].

Usually the construction of measures on infinite dimensional spaces requires the use of a convenient, distinguished class of functions on the space, partly to define the σ-field and partly to provide a means for formulating estimates in some approximation scheme. For example, if the space is linear then the linear functionals play such a role. (See any proof of Kolmogorov's theorem, Minlos' theorem, Sazonov's theorem [26], Levy continuity theorem [24], $P(\phi)_2$ measures [22, 42].) If $G = U(1)$ then $\mathcal{M}^\infty$ is a linear space. For each real-valued function f in $C_c^\infty(R^n)$ and each i and j the linear functional

$$A \to \int_{R^n} F_{ij}(x)\, f(x)\, dx \tag{1.3}$$

is invariant under the action (1.2) of the gauge group $C^\infty(R^n; G)$, hence defines a function on the quotient space $\mathcal{M}^\infty$. Moreover, these linear functionals suffice to determine completely a relevant completion $\mathcal{M}$ on which μ lives as a probability measure, as well as the appropriate σ-field in $\mathcal{M}$. As already mentioned, this can be deduced from Minlos' theorem.

However, if G is not commutative this procedure breaks down: First, the (nonlinear) function (1.3) is not gauge invariant since (1.2) induces the map $F \to F^g$ with $F^g = g^{-1}(x) F(x) g(x)$. Second, the space $\mathcal{M}^\infty$ should properly be regarded as some kind of "curved" infinite dimensional manifold since it has been shown [36, 43] to have nontrivial cohomology, at least if $G = SU(2)$ and R^n is replaced by S^4. A commonly used substitute for (1.3), when G is noncommutative, is determined by a closed curve σ in R^n; if $P(\sigma, A)$ is the operator on $\mathbb{C}^N$ consisting of parallel translation around σ by the $\mathcal{G}$ valued C^∞ connection form A then $W(\sigma, A) :=$ trace $P(\sigma, A)$ is gauge invariant, hence defines a function on $\mathcal{M}^\infty$. These very natural functions (after a suitable normalization) are widely regarded in the physics literature as appropriate candidates for the "convenient, distinguished" functions referred to at the beginning of this paragraph. Indeed they have fairly direct physical significance [45]. Unfortunately they appear to be too singular, as functions of A, to extend to a set of "μ measure one" in any completion $\mathcal{M}$ on which the hoped for measure μ is supported, at least in four space–time dimensions. A statement of this sort takes on meaning by testing it in the commutative case, $G = U(1)$, where μ is just a Gaussian measure. The functions $W(\sigma, \cdot)$ are in fact so singular that it seems unlikely that even averaging over loops (form $\int_{\text{loops}} :W(\sigma, \cdot): dm(\sigma)$, where $:W(\sigma, \cdot):$ denotes a normalized $W(\sigma, \cdot)$ and m is some measure on a loop space) will produce a μ a.e. defined function on $\mathcal{M}$. See [20, Eq. (3.4)] for a discussion of this. The functions $:W(\sigma, \cdot):$ nevertheless can be and have been taken as the basic objects in an appealing set of axioms based on multiloop functions $\{S_k(\sigma_1, \dots, \sigma_k)\}_{k=1}^\infty$, which would be informally related to μ via the equation $S_k(\sigma_1, \dots, \sigma_k) = \int_{\mathcal{M}} \pi^k{}_{j=1} :W(\sigma_j): \, d\mu$ if the measure μ actually existed. See [41]. Progress has been made by Balaban *et al.* [7–10] and Federbush [17, 18] toward proving the existence of a system S_k satisfying these axioms in three space–time dimensions. If this program should succeed it would allow one to construct quantum fields without constructing the measure μ at all. However, their powerful techniques are not tied to this formulation.

The objective of this work is to explore a different set of gauge invariant functions, which seem to offer a reasonable expectation of usefulness for defining and constructing the Euclidean Yang–Mills measure μ (1.1). Let us recall the holonomy theorem of Ambrose and Singer [3]. Denote by $\pi: E \to M$ a vector bundle over a C^∞ finite dimensional manifold M. Fix a connection on E. Pick a point p in M and let σ be a piecewise C^∞ curve in M starting at p and ending at a point x. Write $P(\sigma)$ for parallel translation in E and write K for the curvature 2-form of the connection. If α and β are tangent vectors to M at x then $K\langle \alpha, \beta \rangle$ is an operator on the fiber, $\pi^{-1}(x)$, while

$$L(\sigma)\langle \alpha, \beta \rangle := P(\sigma)^{-1} K\langle \alpha, \beta \rangle P(\sigma) \tag{1.4}$$

is an operator on the fiber $\pi^{-1}(p)$. Keep p fixed and vary σ, α, and β, obtaining a family of linear operators, $L(\sigma)\langle\alpha, \beta\rangle$, on $\pi^{-1}(p)$. Denote by $\mathscr{L}$ the linear span of this family of operators, and by H_0 the restricted holonomy group of the connection at p. ($H_0 = \{P(\sigma)$: σ is a closed, PC^∞, null homotopic curve beginning at $p\}$.)

THEOREM (Ambrose and Singer). *$\mathscr{L}$ is the Lie algebra of H_0.*

We are primarily interested in the case $M = R^n$, $E = R^n \times \mathbb{C}^N$. The connection is given by a globally defined C^∞ connection form A with values in the Lie algebra, $\mathscr{G}$ of a Lie subgroup G of $U(N)$. The immediate significance of the Ambrose–Singer theorem is that the path dependent 2-forms $L(\sigma)$ contain the same amount of physics as do the holonomy group elements. We shall prove a kind of non-Abelian Stokes theorem (Corollary 2.9) which makes their relationship explicit. Note that if $G = U(1)$ then $L(\sigma)\langle e_i, e_j\rangle = F_{ij}(x)$, wherein x is the endpoint of σ and $e_1,..., e_n$ is the standard basis of R^n. Thus the path dependent 2-forms $L(\sigma)$ are generalizations of the curvature ("field strengths") F_{ij} to the noncommutative case. But it makes conceptual sense to average these objects, unlike the curvature itself, even when G is not commutative. To be more precise, if m is a finite measure on some space of (not necessarily closed) curves σ beginning at a fixed point p, then the operator $L_{ij}(m) := \int L(\sigma)\langle e_i, e_j\rangle \, dm(\sigma)$ acts on the fiber at p. (That is, $L_{ij}(m) \to g(p)^{-1} L_{ij}(m)\, g(p)$ under the gauge transformation (1.2).) Consequently trace[exp $L_{ij}(m)$] is fully gauge invariant. Moreover it reduces in the commutative case $G = U(1)$ to $\exp[\int_{R^n} F_{ij}(x) f(x)\, dx]$, wherein f is the density of the endpoint distribution of m. This is, therefore, an a.e. defined measurable function on the Gaussian measure space discussed above, if m, and hence f, is reasonable. "Reasonable" measures are plentiful. The functions $A \to \operatorname{trace}[\exp L_{ij}(m)]$, $A \to \operatorname{trace}(L_{ij}(m)\, L_{kr}(m'))$, etc., (a) are gauge invariant for general G and (b) reduce, if $G = U(1)$, to a.e. defined measurable functions on the Gaussian measure space. No other functions satisfying (a) and (b) are known to the author. It is proposed to use the $L_{ij}(m)$ as the basis for studying quantized Yang–Mills fields in both the Euclidean and Minkowski regions. It was with this in mind that the work [23] was undertaken, with convergence of the lattice theory expressed in terms of the curvature F_{ij}, rather than in terms of the normalized holonomy group functions $:W(\sigma, \cdot):$. The objects $L_{ij}(\sigma)$ have already been studied in the physics literature since the work of Birula [11] and Mandelstam [28–30].

$L(\sigma)\langle\alpha, \beta\rangle$ can be defined geometrically as a limit as ε decreases to zero of $\varepsilon^{-2}(P(\gamma_\varepsilon) - 1)$, where γ_ε is an "infinitesimal lasso" consisting of the curve σ followed by a small rectangular loop of side ε in the α, β plane, followed by σ^{-1}. Accordingly we shall call $L(\sigma)$ the lasso form associated to the connection form A. It will be necessary to consider a more general object. Choose the initial point p to be the origin of R^n and let $\mathscr{P}$ denote the space of piecewise C^∞ curves in R^n starting at the origin and parametrized by [0, 1]. We define a path 2-form to be a smooth function h on $\mathscr{P}$ whose value $h(\sigma)$ is a $\mathscr{G}$ valued 2-form at the endpoint $\sigma(1)$. See Section 2 for the precise definition of smooth. Among the path 2-forms, for example, are the lasso forms. Question: Which path 2-forms are lasso forms? More explicitly, given a path 2-form h, when does there exist a connection form A such that h is the lasso form for A? Our main theorem gives a necesary and sufficient condition for a path 2-form to be a lasso form. We give five equivalent versions of this condition (Theorems 3.4, 3.13, 4.5, and Corollary 4.12). This amounts to a kind of noncommutative Poincaré lemma, because if $G = U(1)$ it reduces to the statement that a 2-form F on R^n is the exterior derivative of a 1-form A if and only if $dF = 0$. It should be noted however that if G is not commutative then the direct analog of $dF = 0$ is the Bianci identity $D_A F = 0$, which requires knowledge of the connection form A in order to form the covariant exterior derivative. Thus it does not make intrinsic sense to formulate Poincare's lemma directly in terms of the curvature form F, because the expected closedness condition cannot be formulated in terms of F alone. We shall see, on the other hand, that Bianci's identity can be formulated intrinsically for a path 2-form h and provides a necessary but not sufficient condition to ensure that h is a lasso form. See Section 4. It will also be shown (Corollary 3.5) that the restricted gauge orbit space, $\{C^\infty \mathscr{G}$ valued 1-forms on $R^n\}/\{g \in C^\infty(R^n; G)$: $g(0) = 1\}$, is in one to one correspondence with the set of lasso forms. This is a noncommutative generalization of Poincaré's lemma for 1-forms. So modulo one degree of gauge freedom we shall identify the gauge orbit space with a surface in the linear space of path 2-forms.

Wu and Yang [46] have pointed out that when G is not commutative the curvature F_{ij} on R^n is not sufficient to determine its connection form up to gauge equivalence, even locally. This has been discussed extensively in the physics literature. (See [33] and its bibliography.) A lasso form L_{ij}, is also a field-strength-like object, but is not subject to this "gauge field copy" problem on R^n (or on any simply connected manifold) in virtue of Corollary 3.5. This, together with the fact that lasso forms can be characterized among path 2-forms by a simple equation, cf. Eqs. (3.1) and (3.2), means that the lasso forms can be taken as complete data for a classical pure Yang–Mills field. We shall use this in Section 5 to show that the equations of Birula [11] and Mandelstam [30] constitute a precise generalization of Maxwell's equations to nonabelian gauge fields.

The present work is written in the C^∞ category. It is another step to extend these notions to generalized path 2-forms, which will be necessary if one wishes to seek a support space for the Euclidean Yang–Mills measure μ (1.1). It is hoped to address this second step in a future work.

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2. Functional Derivatives of Lasso Forms

Denote by $\mathscr{P}$ the set of piecewise $C^\infty(PC^\infty)$ functions σ: $[0, 1] \to R^n$ for which $\sigma(0) = 0$. By definition σ is in $\mathscr{P}$ if it is continuous, if $\sigma(0) = 0$, and if for some partition, $0 = t_0 < t_1 < \cdots < t_k = 1$ (the irregular points of σ), σ is of class C^∞ on each closed interval $[t_{j-1}, t_j]$. Write $\dot{\sigma}(t) = d\sigma(t)/dt$ wherever the derivative exists and put

$$\|\sigma\| = \int_0^1 |\dot{\sigma}(t)|\, dt \tag{2.1}$$

for σ in $\mathscr{P}$. Then $\mathscr{P}$ is a normed vector space in this norm.

Let G be a Lie group of unitary operators on $\mathbb{C}^N$ and denote by $\mathscr{G}$ its Lie algebra, which we identify with operators on $\mathbb{C}^N$. We consider in this section a C^∞ 1-form A on R^n with values in $\mathscr{G}$. Denote by F its curvature, which we may define by $F = dA + A \wedge A$. Thus if $A(x) = \sum_{i=1}^n A_i(x)\, dx^i$ then $F(x) = \sum_{i<j} F_{ij}(x)\, dx^i \wedge dx^j$ with $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x) + [A_i(x), A_j(x)]$. We adopt the conventions $F(x)\langle \alpha, \beta \rangle = \sum_{i,j} F_{ij}(x)\, \alpha_i \beta_j$ for α and β in R^n and $\langle A(x), \alpha \rangle = A(x) \cdot \alpha = \sum_i A_i(x)\, \alpha_i$. If ψ: $R^n \to \mathscr{G}$ is differentiable its covariant derivative in the direction α is given (when ψ is regarded as a section of $R^n \times \operatorname{End} \mathbb{C}^N$) by

$$(D_\alpha \psi)(x) = (\partial_\alpha \psi)(x) + [A(x) \cdot \alpha, \psi(x)]. \tag{2.2}$$

If σ is a piecewise C^1 curve parametrized by $[0, 1]$ then parallel translation along σ is defined, as usual, by the solution $g(t)$ to the initial value problem

$$dg(t)/dt = -(A(\sigma(t)) \cdot \dot{\sigma}(t))\, g(t),\ g(0) = 1. \tag{2.3}$$

At points t_j, where $\dot{\sigma}$ does not exist $g(t)$ is determined by continuity. We write $P(\sigma) = g(1)$. Then $P(\sigma)$ is in G. We define $L(\sigma)$ now as the chart version of (1.4). That is,

$$L(\sigma)\langle \alpha, \beta \rangle = P(\sigma)^{-1}\, F(\sigma(1))\langle \alpha, \beta \rangle\, P(\sigma) \tag{2.4}$$

for σ in $\mathscr{P}$. $L(\sigma)$ is a 2-form at the endpoint of σ with values in $\mathscr{G}$. As explained in the introduction, we call L the lasso form for A. A will be fixed throughout this section. It is clear from the geometrical interpretation of parallel translation that when A is viewed as a connection form on the vector bundle $R^n \times \mathbb{C}^N$ over R^n, $L(\sigma)\langle\alpha, \beta\rangle$ is an operator on the fiber over the origin.

For $0 \leqslant r \leqslant 1$ we write $\sigma^r(t) = \sigma(rt)$ for $0 \leqslant t \leqslant 1$. For any bounded measurable function u: $[0, 1] \to R^n$ and σ in $\mathscr{P}$ define

$$B(\sigma, u) = \int_0^1 L(\sigma^r) < \dot{\sigma}(r), u(r) > dr. \tag{2.5}$$

It is well known that $P(\sigma)$ is independent of the parametrization of σ. Hence $P(\sigma^r) = g(r)$ (cf. Eq. (2.3)) and therefore $L(\sigma^r)\langle\alpha, \beta\rangle$ is a continuous function of r. So the integral (2.5) exists.

To conclude this introduction of notation let us recall the notion of continuous Frechet differentiability in our context.

DEFINITION 2.1. A function f on $\mathscr{P}$ with values in a normed vector space V is of class C^1 if for each σ in $\mathscr{P}$ and u in $\mathscr{P}$ the derivative

$$(\partial_u f)(\sigma) = df(\sigma + su)/ds|_{s=0}$$

exists, is linear and continuous in u as an operator from $\mathscr{P}$ to V and $\sigma \to (\partial_. f)(\sigma)$ is continuous from $\mathscr{P}$ to the space $\mathscr{B}(\mathscr{P}, V)$ of bounded linear operators; f is of class C^{k+1} if it is of class C^k and $\partial_. f$ is of class C^k; f is of class C^∞ on $\mathscr{P}$ if it is of class C^k for $k = 1, 2,\ldots$. A function on $\mathscr{P}$ or on R^n will be called smooth if it is of class C^∞.

THEOREM 2.2. *$P(\sigma)$, $L(\sigma)$, and $B(\sigma, \cdot)$ are of class C^∞ on $\mathscr{P}$ as functions of σ, the first as a function to operators on $\mathbb{C}^N$, the second as a function to $\Lambda^2(R^n) \otimes \mathscr{G}$, and the third as a function to $\mathscr{B}(\mathscr{P}, \mathscr{G})$. Moreover for u in $\mathscr{P}$,*

$$(\partial_u P)(\sigma) = P(\sigma)\, B(\sigma, u) - (A(\sigma(1)) \cdot u(1))\, P(\sigma) \tag{2.6}$$

and

$$\begin{aligned}(\partial_u L)(\sigma)\langle\alpha, \beta\rangle = {}& [L(\sigma)\langle\alpha, \beta\rangle, B(\sigma, u)] \\ & + P(\sigma)^{-1} (D_{u(1)}F)(\sigma(1))\langle\alpha, \beta\rangle\, P(\sigma).\end{aligned} \tag{2.7}$$

LEMMA 2.3. *The map $\sigma \to P(\sigma)$ is continuous from $\mathscr{P}$ to $\mathscr{G}$ and uniformly continuous on bounded sets in $\mathscr{P}$.*

Proof. Let σ and γ be curves in $\mathscr{P}$ and denote by $f(t)$ and $g(t)$ the corresponding parallel translation operators up to time t. Then from (2.3) we get

$$dg(t)^{-1} f(t)/dt = g(t)^{-1} (A(\gamma(t)) \cdot \dot{\gamma}(t) - A(\sigma(t)) \cdot \dot{\sigma}(t))\, f(t).$$

Since $f(t)$ and $g(t)$ are unitary we have

$$\begin{aligned}\|P(\sigma)-P(\gamma)\| &= \|P(\gamma)^{-1}P(\sigma)-1\| \\ &= \|g(1)^{-1}f(1)-1\| \\ &= \left\|\int_0^1 g(t)^{-1}\,(A(\gamma(t))\cdot\dot\gamma(t)-A(\sigma(t))\cdot\dot\sigma(t))\,f(t)\,dt\right\| \\ &\leqslant \int_0^1 \|\{A(\gamma(t))-A(\sigma(t))\}\cdot\dot\gamma(t)\|\,dt \\ &\quad+\int_0^1 \|A(\sigma(t))\cdot(\dot\gamma(t)-\dot\sigma(t))\|\,dt \\ &\leqslant \|\gamma\|\sup_t\|A(\gamma(t))-A(\sigma(t))\|+\|\gamma-\sigma\|\sup_t\|A(\sigma(t))\|.\end{aligned}$$

Since $|\sigma(t)-\gamma(t)|\leqslant\|\sigma-\gamma\|$ and $|\sigma(t)|\leqslant\|\sigma\|$ for all t in $[0,1]$ and A is uniformly continuous on bounded sets the lemma follows.

LEMMA 2.4. *Let $\gamma\colon(-\varepsilon,\varepsilon)\times[a,b]\to R^n$ be of class C^∞. Define $g(s,t)$ for $|s|<\varepsilon$ and t in $[a,b]$ to be the solution to the initial value problem*

$$\partial g(s,t)/\partial t=-(A(\gamma(s,t))\cdot\gamma_t(s,t))\,g(s,t),\ g(s,a)=1, \tag{2.8}$$

where $\gamma_t(s,t)=\partial\gamma(s,t)/\partial t$. Then

$$\begin{aligned}&\partial g(s,b)/\partial s\,|_{s=0}\\ &\quad= g(0,b)\int_a^b g(0,t)^{-1}\,F(\gamma(0,t))\langle\gamma_t(0,t),\gamma_s(0,t)\rangle\,g(0,t)\,dt\\ &\qquad+g(0,b)(A(\gamma(0,a))\cdot\gamma_s(0,a))\\ &\qquad-(A(\gamma(0,b))\cdot\gamma_s(0,b))\,g(0,b).\end{aligned} \tag{2.9}$$

Proof. Let $k(s,t)$ be the unique solution for each t to the initial value problem

$$\partial k(s,t)/\partial s=-(A(\gamma(s,t))\cdot\gamma_s(s,t))\,k(s,t),\ k(0,t)=1, \tag{2.10}$$

where $\gamma_s(s,t)=\partial\gamma(s,t)/\partial s$. Then $k(s,t)$ is of class C^∞ on $(-\varepsilon,\varepsilon)\times[a,b]$. Put $\psi(s,t)=g(0,t)^{-1}\ k(s,t)^{-1}\ g(s,t)\ k(s,a)$. Geometrically $\psi(s,t)$ is parallel translation around a long narrow twisted "rectangle" beginning at $\gamma(0,a)$. Put $V(s,t)=A(\gamma(s,t))\cdot\gamma_t(s,t)$ and $W(s,t)=A(\gamma(s,t))\cdot\gamma_s(s,t)$. Then

$$\begin{aligned}\partial\psi(s,t)/\partial s=g(0,t)^{-1}\,\{&k(s,t)^{-1}\,W(s,t)\,g(s,t)\,k(s,a)\\ &+k(s,t)^{-1}\,(g_s(s,t)\,k(s,a)+g(s,t)\,k_s(s,a))\}.\end{aligned}$$

Hence

$$\partial\psi(s, t)/\partial s|_{s=0} = g(0, t)^{-1} W(0, t)\, g(0, t) + g(0, t)^{-1}\, g_s(0, t) + k_s(0, a).$$

Therefore

$$\begin{aligned}
&\partial^2\psi(s, t)/\partial t\,\partial s|_{s=0} \\
&\quad = g(0, t)^{-1}\,\{[V(0, t), W(0, t)] + W_t(0, t)\}\, g(0, t) \\
&\qquad + g(0, t)^{-1}\, V(0, t)\, g_s(0, t) \\
&\qquad + g(0, t)^{-1}\, \partial g_s(0, t)/\partial t.
\end{aligned}$$

But $\partial g_s(0, t)/\partial t = \partial g_t(s, t)/\partial s|_{s=0}$, since $g(s, t)$ is of class C^∞ on $(-\varepsilon, \varepsilon) \times [a, b]$. So $\partial g_s(0, t)/\partial t = -\partial(V(s, t)\, g(s, t))/\partial s|_{s=0} = -V_s(0, t)$ $g(0, t) - V(0, t)\, g_s(0, t)$. Hence

$$\begin{aligned}
&\partial^2\psi(s, t)/\partial t\,\partial s|_{s=0} \\
&\quad = g(0, t)^{-1}\,\{[V(0, t), W(0, t)] + W_t(0, t) - V_s(0, t)\}\, g(0, t).
\end{aligned}$$

Since $\gamma_{st} = \gamma_{ts}$ we may compute in a straightforward way that the expression in braces equals $F(\gamma(0, t)) < \gamma_t(0, t), \gamma_s(0, t)\rangle$. Thus

$$\begin{aligned}
&\partial^2\psi(s, t)/\partial t\,\partial s|_{s=0} \\
&\quad = g(0, t)^{-1}\, F(\gamma(0, t))\langle\gamma_t(0, t), \gamma_s(0, t)\rangle\, g(0, t). \qquad (2.11)
\end{aligned}$$

Now from the definition of ψ one sees that $\psi(s, a) = 1$ for all s in $(-\varepsilon, \varepsilon)$. Therefore $\psi(s, b) = 1 + \int_a^b (\partial\psi(s, t)/\partial t)\, dt$. The integrand is of class C^∞ on $(-\varepsilon, \varepsilon) \times [a, b]$. So we may write

$$\partial\psi(s, b)/\partial s|_{s=0} = \int_a^b (\partial^2\psi(s, t)/\partial t\,\partial s)|_{s=0}\, dt. \qquad (2.12)$$

Moreover the definition of ψ also gives

$$g(s, b) = k(s, b)\, g(0, b)\, \psi(s, b)\, k(s, a)^{-1}.$$

Thus, since $\psi(0, b) = 1$, we have

$$\begin{aligned}
\partial g(s, b)/\partial s|_{s=0} = &-(A(\gamma(0, b)) \cdot \gamma_s(0, b))\, g(0, b) \\
&+ g(0, b)\, \partial\psi(s, b)/\partial s|_{s=0} \\
&+ g(0, b)(A(\gamma(0, a)) \cdot \gamma_s(0, a)). \qquad (2.13)
\end{aligned}$$

Combining (2.11), (2.12), and (2.13) gives (2.9).

LEMMA 2.5. *For σ and u in $\mathscr{P}$, Eq. (2.6) holds.*

Proof. Denote by $\{t_j\}_{j=1}^{k-1}$ the union of the irregular points for σ and u with $t_0 \equiv 0 < t_1 < t_2 < \cdots < t_k \equiv 1$. Let $\gamma_j(s, t) = \sigma(t) + su(t)$ for $t_{j-1} \leqslant t \leqslant t_j$. Then γ_j is of class C^∞. Let $g_j(s, t)$ denote parallel translation from t_{j-1} to t along $\gamma_j(s, t)$ as in the preceding lemma with $a = t_{j-1}$ and $b = t_j$ and $j = 1, 2, \ldots, k$. Then

$$P(\sigma + su) = g_k(s, t_k) \cdots g_1(s, t_1). \tag{2.14}$$

We need only apply the product rule for differentiation as follows. Write $P_j = g_j(0, t_j)$, $j = 1, \ldots, k$, and let $g(t)$ be the solution to (2.3). Parallel translation from t_{j-1} to t along σ can be accomplished by first parallel translating from t_{j-1} back along σ to 0 and then from 0 to t. That is,

$$g_j(0, t) = g(t)(P_{j-1} \cdots P_1)^{-1}.$$

Put $R_j = A(\sigma(t_j)) \cdot u(t_j)$. Note that $R_0 = 0$. Then Lemma 2.4 states that

$$\begin{aligned} &dg_j(s, t_j)/ds\,|_{s=0} \\ &\quad = P_j R_{j-1} - R_j P_j \\ &\qquad + P_j P_{j-1} \cdots P_1 \int_{t_{j-1}}^{t_j} g(t)^{-1} F(\sigma(t))\langle \dot\sigma(t), u(t)\rangle \\ &\qquad \times g(t)\, dt (P_{j-1} \cdots P_1)^{-1}. \end{aligned} \tag{2.15}$$

Differentiating (2.14) with respect to s at $s = 0$ gives

$$dP(\sigma + su)/ds\,|_{s=0} = \sum_{j=1}^{k} P_k \cdots P_{j+1} (dg_j(s, t_j)/ds)\Big|_{s=0} P_{j-1} \cdots P_1.$$

Substituting (2.15) into this equation one sees that all the boundary terms cancel except the last, which is $-R_k P(\sigma)$, while the integral terms add up to $P(\sigma) \int_0^1 g(t)^{-1} F(\sigma(t))\langle \dot\sigma(t), u(t)\rangle\, g(t)\, dt$, which is $P(\sigma)\, B(\sigma, u)$.

LEMMA 2.6. *For each σ and u in $\mathscr{P}$, Eq.* (2.7) *holds.*

Proof. (2.6) implies that

$$dP(\sigma + su)^{-1}/ds\,|_{s=0} = -B(\sigma, u)\, P(\sigma)^{-1} + P(\sigma)^{-1} A(\sigma(1)) \cdot u(1).$$

Hence

$$\begin{aligned} &(\partial_u L)(\sigma)\langle \alpha, \beta\rangle \\ &\quad = \partial_u (P(\sigma)^{-1} F(\sigma(1))\langle \alpha, \beta\rangle P(\sigma)) \end{aligned}$$

$$
\begin{aligned}
&= \{-B(\sigma, u)\, P(\sigma)^{-1} + P(\sigma)^{-1} A(\sigma(1)) \cdot u(1)\}\, F(\sigma(1))\langle \alpha, \beta \rangle\, P(\sigma) \\
&\quad + P(\sigma)^{-1} (\partial_{u(1)} F)(\sigma(1))\langle \alpha, \beta \rangle\, P(\sigma) \\
&\quad + P(\sigma)^{-1} F(\sigma(1))\langle \alpha, \beta \rangle \{P(\sigma)\, B(\sigma, u) - A(\sigma(1)) \cdot u(1)\, P(\sigma)\} \\
&= [L(\sigma)\langle \alpha, \beta \rangle, B(\sigma, u)] \\
&\quad + P(\sigma)^{-1} \{(\partial_{u(1)} F)(\sigma(1))\langle \alpha, \beta \rangle \\
&\quad + [A(\sigma(1) \cdot u(1), F(\sigma(1))\langle \alpha, \beta \rangle]\}\, P(\sigma)
\end{aligned}
$$

which is Eq. (2.7).

LEMMA 2.7. *The map $r, \sigma \to \sigma^r$ is continuous from $[0, 1] \times \mathscr{P}$ to $\mathscr{P}$.*

Proof. $\int_0^1 |rf(rt) - sf(st)|\, dt$ goes to zero as $|r - s| \to 0$ if f is a continuous function from $[0, 1]$ into R^n, and hence, by a standard approximation argument, also if f is merely integrable. Thus $\|\gamma^r - \gamma^s\| \to 0$ as $|r - s| \to 0$ for any point γ in $\mathscr{P}$. Since $\|\sigma^r\| \leqslant \|\sigma\|$ we therefore have $\|\sigma^r - \gamma^s\| \leqslant \|\sigma^r - \gamma^r\| + \|\gamma^r - \gamma^s\| \leqslant \|\sigma - \gamma\| + \|\gamma^r - \gamma^s\|$ which goes to zero as $\sigma \to \gamma$ and $r \to s$.

Remark. It is clear from the proof that if $\mathscr{P}$ were replaced by its completion, the set of all absolutely continuous functions σ on $[0, 1]$ to R^n with $\sigma(0) = 0$, the lemma would still hold. In fact all of our results extend to the completion. The quadratic norm, $\|\sigma\|_2 := (\int_0^1 |\dot{\sigma}(t)|^2\, dt)^{1/2}$ would do just as well. In this case $\mathscr{P}$ is known to be a Hilbert manifold [15, 16, 37] and may be a natural setting for some purposes related to the present work.

LEMMA 2.8. *$L(\sigma)$ and $B(\sigma, \cdot)$ are continuous functions of σ on $\mathscr{P}$ and are bounded on bounded sets in $\mathscr{P}$.*

Proof. $L(\sigma)\langle \alpha, \beta \rangle = P(\sigma)^{-1} F(\sigma(1))\langle \alpha, \beta \rangle\, P(\sigma)$ is a product of three continuous functions and since $|\sigma(1)| \leqslant \|\sigma\|$ the middle factor is bounded on bounded sets. The operator $u \to B(\sigma, u)$ is uniformly bounded on bounded sets in $\mathscr{P}$ because

$$
\begin{aligned}
\|B(\sigma, u)\| &= \left\| \int_0^1 \sum_{i,j} L_{ij}(\sigma^r)\, \dot{\sigma}_i(r)\, u_j(r)\, dr \right\| \\
&\leqslant \sum_{i,j} (\sup_r \|L_{ij}(\sigma^r)\|) \int_0^1 |\dot{\sigma}_i(r)|\, dr \sup_r |u_j(r)| \\
&\leqslant \left(\sum_{i,j} \sup_r \|L_{ij}(\sigma^r)\| \right) \|\sigma\|\, \|u\|.
\end{aligned}
$$

The continuity of $\sigma \to B(\sigma, \cdot)$ from $\mathscr{P}$ to the set $\mathscr{B}(\mathscr{P}, \mathscr{G})$, of bounded operators on $\mathscr{P}$ to $\mathscr{G}$ follows from a similar estimate of $\|B(\sigma, u) - B(\gamma, u)\|$. In these inequalities the norm on $\mathscr{G}$-valued functions is the operator norm on operators on $\mathbb{C}^N$.

Proof of Theorem 2.2 Lemmas 2.5 and 2.6 show that the directional derivatives of P and L exist and are given by (2.6) and (2.7), respectively. Lemma 2.8 shows that these directional derivatives are bounded linear functions of u and are continuous in σ. Hence P and L are of class C^1. Moreover equations (2.5), (2.6), and (2.7) show that $\partial_u P$ and $\partial_u L$ may be expressed in terms of P and L again in a smooth way. An induction proof on the statement that P, L, and B are of class C^k with uniformly bounded derivatives up to order k on bounded sets in $\mathscr{P}$ will show that these functions are infinitely differentiable. A key point is the differentiability of $B(\sigma, u)$, which enters as a factor in each successive derivative. We limit the remainder of the proof to a discussion of this differentiability. Since $dL((\sigma + su)^r)/ds|_{s=0} = dL(\sigma^r + su^r)/ds|_{s=0} = (\partial_{u^r} L)(\sigma^r)$ we have

$$\partial_u B(\sigma, v) = \int_0^1 (\partial_{u^r} L)(\sigma^r)\langle \dot{\sigma}(r), v(r)\rangle \, dr$$
$$+ \int_0^1 L(\sigma^r)\langle \dot{u}(r), v(r)\rangle \, dr.$$

Using the L^1 norm on $\dot{u}$ and the sup norm on v we see that the second term is a bounded bilinear form in u and v and continuous at σ (cf. Lemma 2.8). The first term, by Eq. (2.7) is

$$\int_0^1 [L(\sigma^r)\langle \dot{\sigma}(r), v(r)\rangle, B(\sigma^r, u^r)] \, dr$$
$$+ \int_0^1 P(\sigma^r)^{-1} (D_{u(r)} F)(\sigma(r))\langle \dot{\sigma}(r), v(r)\rangle \, P(\sigma^r) \, dr.$$

Once again we see that this is a bounded bilinear form in u and v, by using the L^1 norm on $\dot{\sigma}$, and the sup norm on v and u, all of which norms are dominated by the corresponding $\mathscr{P}$ norms. Continuity in σ follows from Lemmas 2.3 and 2.7 and 2.8.

COROLLARY 2.9. *If σ and u are in $\mathscr{P}$ and $u(1) = 0$ then*

$$\partial_u P(\sigma) = P(\sigma) \, B(\sigma, u). \tag{2.16}$$

Proof. This is a special case of Eq. (2.6).

Remark 2.10. Equation (2.16) can be considered to be a kind of non-commutative extension of Stokes' theorem. Take $G = U(1)$, the circle group. In this case $L(\sigma)$ depends only on the endpoint of σ since G is commutative: $L(\sigma)=F(\sigma(1))$ and then

$$B(\sigma, u)=\int_0^1 F(\sigma(t))\langle\dot{\sigma}(t), u(t)\rangle\, dt,$$

which is a purely imaginary number. Suppose that $\sigma(s, t)$ is an infinitely differentiable function on $[0, 1]\times[0, 1]$ to R^n. We think of σ as a parametrization of a surface S whose boundary, ∂S, is parametrized in its entirety by the curve $t\to\sigma(1, t)$, $0\leqslant t\leqslant 1$. Specifically, we assume that $\sigma(s, 0)=\sigma(s, 1)=\sigma(0, t)=0$ for all s and t in $[0, 1]$. Thus the closed curves $t\to\sigma(s, t)$ fill out the surface as s varies. Clearly $\sigma_s(s, 0)=\sigma_s(s, 1)=0$ for all s in $[0, 1]$. So by (2.16) parallel translation around the closed curve $t\to\sigma(s, t)$ satisfies

$$dP(\sigma(s, \cdot))/ds=P(\sigma(s, \cdot))\, B(\sigma(s, \cdot), \sigma_s) \tag{2.17}$$

and of course $P(\sigma(0, \cdot))=1$, since $\sigma(0, \cdot)$ is a null curve. But the operators $B(\;)$ commute, so the solution of this system is

$$\begin{aligned} P(\sigma(r, \cdot)) &= \exp\int_0^r B(\sigma(s, \cdot), \sigma_s)\, ds \\ &= \exp\int_0^r\int_0^1 F(\sigma(s, t))\langle\sigma_t, \sigma_s\rangle\, dt\, ds. \end{aligned}$$

For $r=1$ the last double integral is exactly minus the "normal" ($\sigma_s\times\sigma_t$ if $n=3$) component of F over S. On the other hand, by the definition of parallel translation, Eq. (2.3), $P(\sigma(1, \cdot))=\exp-\int_0^1 A(\sigma(1, t)\cdot\sigma_t(1, t)\, dt$. Thus

$$\exp\left[-\int_{\partial S} A\right]=\exp\left[-\int_S F\right]. \tag{2.18}$$

If we replace A by αA, for α real, and hence F by αF and then differentiate (2.18) with respect to α at $\alpha=0$, we get $\int_{\partial S} A=\int_S F$, which is Stokes' theorem for the 1-form A.

The Eq. (2.16) also yields an elementary proof of the Ambrose–Singer theorem on R^n. Here is a simple form.

COROLLARY 2.11 (Ambrose and Singer). *Let A be a C^∞ connection form on R^n with values in the Lie algebra $\mathscr{G}$ of a Lie subgroup G of $U(N)$. Let $\mathscr{L}$ be the linear span of $\{L(\sigma)\langle\alpha, \beta\rangle$: $\sigma\in\mathscr{P}$, $\alpha, \beta\in R^n\}$. Then $\mathscr{L}$ is the Lie algebra of the holonomy group of A at* 0.

Proof. Denote by H the holonomy group of A at 0 and by $\mathscr{H}$ its Lie algebra. Recall that $H=\{P(\sigma)\colon \sigma = \text{closed curve in } \mathscr{P}\}$. As noted in the Introduction, $L(\sigma)\langle\alpha, \beta\rangle$ is a limit of the form $\varepsilon^{-2}(P(\sigma_\varepsilon)-1)$, where σ_ε is a small lasso of area ε^2 and is therefore in $\mathscr{H}$. Hence $\mathscr{L} \subset \mathscr{H}$. This is the usual proof of the easy half of the Ambrose–Singer theorem.

To see that $\mathscr{L} \supset \mathscr{H}$ note first that since $\mathscr{H}$ is finite dimensional so is $\mathscr{L}$. Therefore the operators $B(\sigma, u)$ are in $\mathscr{L}$. Moreover if σ is closed and γ is in $\mathscr{P}$ then Eq. (2.4) shows that $P(\sigma)^{-1}\, L(\gamma)\langle\alpha, \beta\rangle\, P(\sigma) = L(\gamma\cdot\sigma)\langle\alpha, \beta\rangle$, where $\gamma\cdot\sigma$ is the usual product, consisting of σ parametrized on $[0, \frac{1}{2}]$ followed by γ parametrized on $[\frac{1}{2}, 1]$. Hence $\mathscr{L}$ is invariant under the adjoint representation of H on $\mathscr{H}$. In particular, $\mathscr{L}$ is a subalgebra of $\mathscr{H}$. Now if γ is a closed curve and is in $\mathscr{P}$, so is $t \to s\gamma(t)$ and we may put $f(s) = P(s\gamma)$ for $0 \leqslant s \leqslant 1$. Then f is a curve in H with $f(0)=1$ and $f(1) = P(\gamma)$. By Eq. (2.16),

$$f(s)^{-1}\, f'(s) = B(s\gamma, \gamma)$$

which is in $\mathscr{L}$. Hence the curve f is everywhere tangent to the integral submanifold of $\mathscr{L}$ (regarded as left invariant vector fields on H) passing through the identity element of H. Therefore the integral submanifold is all of H. Hence $\mathscr{L} = \mathscr{H}$.

Remark 2.12. The non-Abelian Stokes theorem, Eq. (2.16), has been derived a number of times in the physics literature in various forms, with various degrees of rigor and starting with various different definitions of parallel translation [4, 12, 19, 30, 35, 39]. Besides the defining equation (2.3) one can take the Neumann series solution as a definition [4] or define it as a limit of products (path ordered product). The first statement of a local version of (2.7) (cf. Theorem 4.11) seems to be that by Mandelstam [30, Eq. (4.2)]. In the mathematics literature the 1955 book by Lichnerowicz [27] derives a special case of (2.16) in which $u(t) = \sigma(t)$, $0 \leqslant t \leqslant 1$. This "radial" derivative was used in his proof of the Ambrose–Singer theorem in a special gauge. See [27, Eq. (60.5)]. See also Remark 3.9 below.

The results of this section extend easily in several directions as indicated in the following remarks.

Remark 2.13 (Extension to $G = GL(V)$). The assumption that $G \subset U(N)$ was used only in the proof of Lemma 2.3 and only to simplify one estimate. Theorem 2.4 and its corollaries continue to hold if $\mathbb{C}^N$ is replaced by a finite dimensional normed real or complex vector space V and G by a Lie subgroup of $GL(V)$. The proof of Lemma 2.3 requires a minor modification. Indeed it is elementary to show that the function $g(t)$ in the proof satisfies $\|g(t)\| \leqslant C(A, \|\sigma\|)$ for $0 \leqslant t \leqslant 1$, and for some real valued function $C(A, s)$. The estimate of $\|P(\sigma)-P(\gamma)\|$ must be increased by a factor $C(A, \|\sigma\|)\, C(A, \|\gamma\|)$.

Remark 2.14 (Enlargement of the space $\mathscr{P}$). $\mathscr{P}$ is dense in the space $\bar{\mathscr{P}}$ of absolutely continuous functions σ: $[0, 1] \to R^n$ for which $\sigma(0)=0$, in the norm (2.1). Lemma 2.3 shows that the function $\sigma \to P(\sigma)$ has a unique continuous extension to all of $\bar{\mathscr{P}}$. Therefore so does $L(\sigma)$ and $B(\sigma, u)$, for u in $\bar{\mathscr{P}}$. Since the derivatives $\partial_u P(\sigma)$, $\partial_u L(\sigma)$, and $\partial_u B(\sigma, v)$ are all expressible in terms of P, L, and B it follows that Eqs. (2.6), (2.7), and (2.16) continue to hold for σ and u in $\bar{\mathscr{P}}$. In fact our proof of Theorem 2.2 really shows that P, L, and B are of class C^∞ on $\bar{\mathscr{P}}$.

Remark 2.15 (Extension to manifolds). Let π: $E \to M$ be a vector bundle over a finite dimensional manifold M of class C^∞. Denote by V the fiber, a finite dimensional vector space. Pick a point a in M and let $\mathscr{P}$ be the set of all piecewise C^∞ functions σ from $[0, 1]$ into M such that $\sigma(0)=a$. In the following we fix a C^∞ connection on E and write $P(\sigma)$ for its parallel translation operators. If we put $E_b=\pi^{-1}(b)$ then $P(\sigma)$: $E_a \to E_{\sigma(1)}$ for σ in $\mathscr{P}$. Let K be the curvature 2-form of the connection. If α and β are vector fields on M and α_x denotes the value of α at $x \in M$ then $K\langle \alpha_x . \beta_x \rangle$ is an operator on the fiber E_x. Define

$$L(\sigma)\langle \alpha, \beta \rangle = P(\sigma)^{-1}\, K\langle \alpha_x, \beta_x \rangle\, P(\sigma) \tag{2.19}$$

for σ in $\mathscr{P}$, wherein $x=\sigma(1)$. Then $L(\sigma)\langle \alpha, \beta \rangle$ is an operator on E_a for all σ in $\mathscr{P}$.

$\mathscr{P}$ is no longer a vector space. In order to give meaning to the functional derivatives occurring in Theorem 2.2 and its corollaries it is necessary to give $\mathscr{P}$ the structure of a differentiable manifold (infinite dimensional). The machinery for doing this is well developed [2, 15, 16, 37]. Moreover, one need not use only the very small space $\mathscr{P}$ but one can use, for example, the space $\bar{\mathscr{P}}$ of absolutely continuous functions into M, or the space $\hat{\mathscr{P}}$ consisting of those σ in $\bar{\mathscr{P}}$ with square integrable derivative. (These are chart independent notions.) The key point in converting such a space into a differentiable manifold is to define the tangent space at a point σ as the set of all functions u: $[0, 1] \to TM$ such that $u(t)$ is in $T_{\sigma(t)}M$ and such that u has the same smoothness properties as a generic σ. Thus for the space $\hat{\mathscr{P}}$, u should be absolutely continuous with a square integrable derivative. (This is also a chart independent notion.) In this case $\hat{\mathscr{P}}$ has the structure of a Hilbert manifold in a natural way and the tangent space $T_\sigma \hat{\mathscr{P}}$ may be identified with the set of these functions u. A local diffeomorphism ψ from a neighborhood of 0 in $T_\sigma \hat{\mathscr{P}}$ to a neighborhood of σ can be constructed by choosing a smooth Riemannian metric on M and putting, for small u, $\psi(u)(t)=\exp(\sigma(t), u(t))$, $0<t\leqslant 1$, where $s \to \exp(x, su)$ is the geodesic

through x in the tangent direction $v \in T_x M$. See, e.g., [15; 16; 37, Sect. 13]. The derivative of a function f on $\hat{\mathscr{P}}$ in the direction u at σ may be defined by

$$(\partial_u f)(\sigma) = df(\gamma(s))/ds\,|_{s=0}, \tag{2.20}$$

where $\gamma(s)$ is any C^1 function from $(-\varepsilon, \varepsilon)$ into $\hat{\mathscr{P}}$ with $\gamma(0) = \sigma$ and $\partial\gamma(s, t)/\partial s\,|_{s=0} = u(t)$. ($\partial\gamma/\partial s$ must be interpreted in a $\hat{\mathscr{P}}$ sense.) For example, the curve $\gamma(s)$ given by $\gamma(s, t) = \exp(\sigma(t), su(t))$ is tangent to u at $s = 0$. If the derivative in (2.20) exists in a suitably uniform manner (Frechet derivative) then the usual calculus of derivatives holds. We shall show in the next corollary how the basic equations (2.7) and (2.16) may be derived from Lemma 2.4. We restrict our attention to the simplest case: σ and u are of class PC^∞. Extensions of these equations to the large space $\bar{\mathscr{P}}$ can be deduced as in Remark 2.14.

COROLLARY 2.16. *Let γ: $(-\varepsilon, \varepsilon) \times [0, 1] \to M$ be a continuous function. Assume that γ is of class C^∞ on the sets $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j]$ with $0 = t_0 < t_1 < \cdots < t_k = 1$, and that $\gamma(s, 0) = a$ for $-\varepsilon < s < \varepsilon$. Write $\sigma(t) = \gamma(0, t)$ and $u(t) = \partial\gamma(s, t)/\partial s\,|_{s=0}$. Then for any smooth vector fields α and β on M,*

$$\partial_u(L(\sigma)\langle\alpha, \beta\rangle) = [L(\sigma)\langle\alpha, \beta\rangle, B(\sigma, u)] + P(\sigma)^{-1} D_{u(1)}(K\langle\alpha, \beta\rangle)\, P(\sigma), \tag{2.21}$$

wherein $D_{u(1)}(K\langle\alpha, \beta\rangle)$ denotes the covariant derivative at $\sigma(1)$ in the direction $u(1)$ of the section $K\langle\alpha, \beta\rangle$ of the bundle Hom(E, E). *Here $B(\sigma, u)$, as given by* (2.5), *is an operator on the fiber E_a. If $u(1) = 0$ then*

$$\partial_u P(\sigma) = P(\sigma)\, B(\sigma, u). \tag{2.22}$$

Proof. By increasing the number of division points $\{t_j\}$ we may assume that they are so closely spaced that there exist coordinate patches $U_j \subset M$ such that $\sigma(t)$ is in U_j for $t_{j-1} \leqslant t \leqslant t_j$ and that E trivializes over U_j: $E \mid U_j \cong U_j \times V$. By reducing ε we may also assume that $\gamma(s, t)$ is in U_j for $t_{j-1} \leqslant t \leqslant t_j$ and $|s| < \varepsilon$. Let A_j be the connection form for the given connection relative to the chart $U_j \times V$. Thus $A_j(x)$ is an operator on V for each x in U_j and A_j is of class C^∞ on U_j. For each x in $U_j \cap U_{j+1}$ let $\phi_j(x) \in GL(V)$ be the transition operator. Thus (x, v) in the chart $U_{j+1} \times V$ corresponds to the same point in E_x as $(x, \phi_j(x)\, v)$ does in the chart $U_j \times V$. ϕ_j is of class C^∞ on $U_j \cap U_{j+1}$, $j = 1, \ldots, k-1$. Then

$$A_{j+1}(x) = \phi_j(x)^{-1} A_j(x)\, \phi_j(x) + \phi_j(x)^{-1}\, d\phi_j(x) \tag{2.23}$$

for x in $U_j \cap U_{j+1}$ and $1 \leqslant j \leqslant k-1$. If τ is a curve in U_j we write $P_j(\tau)$: $V \to V$ for the parallel translation operator determined by A_j. Now

$P(\sigma)$ is an operator from $E_{\sigma(0)}$ to $E_{\sigma(1)}$. However with the aid of the first and last charts, $U_1 \times V$ and $U_k \times V$, we may and shall identify $P(\sigma)$ with an operator from V into V. In fact since $\gamma(s, 0) \equiv a$ and $\gamma(s, 1)$ is in U_k for $|s| < \varepsilon$, $P(\gamma(s, \cdot))$ may be identified with an operator from V into V for $|s| < \varepsilon$. It is the derivative of this operator that we shall compute. Let $\gamma_j(s, t) = \gamma(s, t)$ for $t_{j-1} \leqslant t \leqslant t_j$, and $|s| < \varepsilon$. Then $\gamma_j(s, \cdot)$ is a curve in U_j for each s and therefore $P_j(\gamma_j(s, \cdot))$ is well defined as an operator on V. Moreover

$$P(\gamma(s, \cdot)) = P_k(\gamma_k(s, \cdot))\, \phi_{k-1}(\gamma(s, t_{k-1}))^{-1}\, P_{k-1}(\gamma_{k-1}(s, \cdot)) \\ \cdots \phi_1(\gamma(s, t_1))^{-1}\, P_1(\gamma_1(s, \cdot)). \tag{2.24}$$

With the aid of Lemma 2.4 we may express the derivative of each factor $P_j(\gamma_j(s, \cdot))$ with respect to s at $s = 0$ in terms of an integral of lasso forms plus boundary terms. The sum over j of the integral terms combine, just as in the proof of Lemma 2.5 to give $P(\sigma)\, B(\sigma, u)$. Thus

$$dP(\gamma(s, \cdot))/ds\,|_{s=0} = P(\sigma)\, B(\sigma, u) + \text{boundary terms}.$$

But, with $a_j = \sigma(t_j)$, the boundary term in $d\phi_j(\gamma(s, t_j))^{-1}\, P_j(\gamma_j(s, \cdot))/ds\,|_{s=0}$ is

$$-\phi_j(a_j)^{-1} \langle d\phi_j(a_j), u(t_j)\rangle\, \phi_j(a_j)^{-1}\, P_j(\sigma_j) \\ + \phi_j(a_j)^{-1} \{P_j(\sigma_j)\langle A_j(a_{j-1}), u(t_{j-1})\rangle - \langle A_j(a_j), u(t_j)\rangle\, P_j(\sigma_j)\}$$

which equals

$$\phi_j(a_j)^{-1}\, P_j(\sigma_j)\langle A_j(a_{j-1}), u(t_{j-1})\rangle - \langle A_{j+1}(a_j), u(t_j)\rangle\, \phi_j(a_j)^{-1}\, P_j(\sigma_j)$$

by (2.23). Inserting this into the derivative of (2.24) we see that all the boundary terms cancel except the last, leaving

$$dP(\gamma(s, \cdot))/ds\,|_{s=0} = P(\sigma)\, B(\sigma, u) - A_k(\sigma(1)) \cdot u(1)\, P(\sigma). \tag{2.25}$$

If $u(1) = 0$ then (2.25) reduces to (2.22). (Note that in (2.25) the operators $P(\sigma)$ and $B(\sigma, u)$ are operators on V while in (2.22) these same symbols represent operators on fibers!) The derivation of (2.21) from (2.25) procedes exactly as in Lemma 2.6 but with A replaced by A_k.

COROLLARY 2.17 (Ambrose and Singer). *Let $\pi: E \to M$ be a vector bundle over a finite dimensional manifold M of class C^∞. Fix a smooth connection on E and pick a point a in M. Let $\mathscr{L}$ be the linear span of $\{L(\sigma)\langle \alpha, \beta\rangle$: $\sigma \in \mathscr{P}$, α, $\beta \in T_{\sigma(1)}M\}$. Then $\mathscr{L}$ is the Lie algebra of the restricted holonomy group at a.*

Proof. The proof is the same as that of Corollary 2.11 except that the homotopy $s, t \to s\gamma(t)$ of the PC^∞ closed curve γ with the null curve used there must be replaced. If γ is a closed curve in M beginning and ending at a and of class PC^∞ and is, moreover, homotopic to the null curve (with endpoints fixed in the homotopy) then it is also homotopic to the null curve via a piecewise smooth homotopy $\gamma(s, t)$ [34], in such a way that the map $s \to \gamma(s, \cdot)$ from $[0, 1]$ into $\mathscr{P}$ is continuous and piecewise C^1. One can now apply the non-Abelian Stokes theorem, Eq. (2.22), just as in the proof of Corollary 2.11.

Remark 2.18. The results of this section can be formulated in terms of principal bundles. The map $\sigma \to P(\sigma)$ is now a map from curves to (horizontal) curves and the Eqs. (2.6), (2.7), and (2.16) become simple statements about the differential of this map from one infinite dimensional manifold to another.

3. Poincaré Lemma for Path 2-Forms

Definition 3.1. A (differential) path k-form at 0 in R^n is a function h on $\mathscr{P}$ whose value, $h(\sigma)$, at a point σ in $\mathscr{P}$ is a k-form on R^n at the endpoint of σ. The k-form may take its values in a vector space.

We are primarily interested in path 2-forms h with values in the Lie algebra $\mathscr{G}$ discussed in Section 2. Thus for vectors α and β in R^n and σ in $\mathscr{P}$ $h(\sigma)\langle\alpha, \beta\rangle$ is in $\mathscr{G}$. For example, a $C^\infty\mathscr{G}$-valued 1-form A on R^n gives rise in a natural way to the lasso form L (defined in Eq. (2.4)) which is a $\mathscr{G}$-valued path 2-form at 0. Not every smooth path 2-form h is a lasso form for some connection form A. The objective of this paper is to characterize those that are.

We denote by $\mathscr{P}_0$ those elements σ in $\mathscr{P}$ such that $\sigma(1) = 0$.

Definition 3.2. *Let h be a $\mathscr{G}$-valued path 2-form on $\mathscr{P}$ of class C^∞. Put*

$$B(\sigma, u) = \int_0^1 h(\sigma^r)\langle\dot{\sigma}(r), u(r)\rangle \, dr \tag{3.1}$$

for σ and u in $\mathscr{P}_0$. Here $\sigma^r(t) = \sigma(rt)$ for $0 \leqslant t \leqslant 1$ and $0 \leqslant r \leqslant 1$; h is called closed *if the associated 1-form B on $\mathscr{P}_0$ has curvature zero, i.e., if for all σ, u, and v in $\mathscr{P}_0$*

$$\partial_u B(\sigma, v) - \partial_v B(\sigma, u) + [B(\sigma, u), B(\sigma, v)] = 0. \tag{3.2}$$

Remark 3.3. By Lemma 2.7, σ^r is jointly continuous in σ and r from $[0, 1] \times \mathscr{P}$ into $\mathscr{P}$ and is linear in σ for each r. Therefore $h(\sigma^r)\langle\alpha, \beta\rangle$ is continuous in r and σ and C^∞ in σ for each r. Moreover $\dot{\sigma}$ is integrable and

u is bounded, so the integral in (3.1) exists and is controlled by the norm on $\mathscr{P}_0 \times \mathscr{P}_0$. Moreover $B(\sigma, u)$ is continuous on $\mathscr{P}_0 \times \mathscr{P}_0$. In fact the map $\sigma \to B(\sigma, \cdot)$ from $\mathscr{P}_0$ to $\mathscr{B}(\mathscr{P}_0, \mathscr{G})$ is infinitely differentiable and the kth derivative is easily seen by induction to be given by

$$\begin{aligned} &\partial_{v_1} \cdots \partial_{v_k} B(\sigma, u) \\ &\quad = \int_0^1 (\partial_{v_{1r}} \cdots \partial_{v_{kr}} h)(\sigma^r)\langle \dot{\sigma}(r), u(r)\rangle \, dr \\ &\qquad + \sum_{j=1}^{k} \int_0^1 (\partial_{v_{1r}} \cdots \hat{\partial}_{v_{jr}} \cdots \partial_{v_{kr}} h)(\sigma^r)\langle \dot{v}_j(r), u(r)\rangle \, dr. \end{aligned}$$

Every term is a bounded multilinear form in $v_1, \ldots, v_k$ in the $\mathscr{P}_0$ norm.

THEOREM 3.4 (Poincaré lemma). *Let h be a $\mathscr{G}$-valued path 2-form on $\mathscr{P}$ of class C^∞. There exists a $\mathscr{G}$-valued 1-form A on R^n of class C^∞ such that h is the lasso form for A if and only if h is closed.*

It is customary to call the group $C^\infty(R^n; G)$ the local gauge group, the subgroup $C_0^\infty(R^n; G) := \{g \in C^\infty(R^n; G) \colon g(0) = 1\}$ the restricted local gauge group, the map (1.2) a restricted gauge transformation if g is in $C_0^\infty(R^n; G)$, and $\{C^\infty \mathscr{G}\text{-valued 1-forms } A \text{ on } R^n\}/C_0^\infty(R^n; G)$ the restricted gauge orbit space.

COROLLARY 3.5. *Denote by L^A the lasso form for A at 0. Then the map $A \to L^A$ defines a one to one correspondence between the restricted gauge orbit space and the space of closed C^∞ path 2-forms at 0.*

Proof of Theorem 3.4. Assume that h is the lasso form for some C^∞ 1-form A on R^n. If σ and u are in $\mathscr{P}_\sigma$ then Eq. (2.16) holds. Since P is a C^∞ function on $\mathscr{P}_0$ by Theorem 2.2, we have $\partial_v \partial_u P(\sigma) = \partial_u \partial_v P(\sigma)$ for u, v, and σ in $\mathscr{P}_0$. By (2.16),

$$\partial_v \partial_u P(\sigma) = P(\sigma)\, B(\sigma, v)\, B(\sigma, u) + P(\sigma)\, \partial_v B(\sigma, u)$$

with a similar identity for $\partial_u \partial_v P(\sigma)$. Subtracting the first identity from the second gives (3.2), since $P(\sigma)$ is in G and may be cancelled.

For the converse, assume now that h is a closed path 2-form on $\mathscr{P}$ of class C^∞. B is defined by (3.1).

LEMMA 3.6. *There exists a unique infinitely differentiable function $\hat{P} \colon \mathscr{P}_0 \to G$ such that $\hat{P}(0) = 1$ and*

$$\partial_u \hat{P}(\sigma) = \hat{P}(\sigma)\, B(\sigma, u) \tag{3.3}$$

for all σ and u in $\mathscr{P}_0$.

Proof. This lemma can be deduced from an infinite dimensional version of Frobenius's theorem applied to the completion of the space $\mathscr{P} \times G$ in a standard way. But we give a direct and short proof based on (2.16). Denote by $\mathscr{R}$ the vector space of PC^∞ functions Γ: $[0, 1] \to \mathscr{P}_0$ such that $\Gamma(0) = 0$. Put $\|\Gamma\| = \int_0^1 \|\dot{\Gamma}(t)\|\, dt$, where $\|\dot{\Gamma}(t)\|$ is given by (2.1). By Remark 3.3, B is a C^∞ 1-form on $\mathscr{P}_0$. Define parallel translation with respect to B by putting $R(\Gamma) = g(1)$, where

$$\dot{g}(t) = -B(\Gamma(t), \dot{\Gamma}(t))\, g(t),\ g(0) = 1. \tag{3.4}$$

We are clearly in a situation analogous to that of Section 2 with R^n, A, $\mathscr{P}$, P replaced by $\mathscr{P}_0$, B, $\mathscr{R}$, R, respectively. But Theorem 2.2 nowhere uses the completeness of R^n or its finite dimensionality in an essential way. We may therefore take over the purely computational conclusions of Theorem 2.2 to the present infinite dimensional context. In particular if $\mathscr{R}_0 = \{\Gamma \in \mathscr{R}: \Gamma(1) = 0\}$ then by Eq. (2.16) (noncommutative Stokes theorem) we have $\partial_U R(\Gamma) = R(\Gamma)\, R(\Gamma)\, C(\Gamma, U)$ for Γ and U in $\mathscr{R}_0$, where

$$\begin{aligned} C(\Gamma, U) = \int_0^1 & R(\Gamma^r)^{-1} \{\text{curvature of } B\} \\ & \times \langle \dot{\Gamma}(r), U(r) \rangle\, R(\Gamma^r)\, dr. \end{aligned}$$

Cf. (2.4) and (2.5). But by assumption the curvature of B is zero. So $\partial_U R(\Gamma) = 0$ for all Γ and U in $\mathscr{R}_0$. In particular if $U = s\Gamma$ this gives $dR(s\Gamma)/ds = 0$. So $R(\Gamma) = R(0) = 1$ for all Γ in $\mathscr{R}_0$. Thus parallel translation for B is path independent. For any point σ in $\mathscr{P}_0$ put $\Gamma(t) = t\sigma$, $0 \leqslant t \leqslant 1$, and define

$$\hat{P}(\sigma) = R(\Gamma)^{-1}. \tag{3.5}$$

Since $\sigma \to \Gamma$ is continuous and linear from $\mathscr{P}_0$ to $\mathscr{R}$ and R is of class C^∞ on $\mathscr{R}$ it follows that $\hat{P}$ is of class C^∞ on $\mathscr{P}_0$. Clearly $\hat{P}(0) = 1$. Moreover by path independence of R, Eq. (3.5) holds for any Γ in $\mathscr{R}$ with $\Gamma(1) = \sigma$. To verify (3.3) for σ and u in $\mathscr{P}_0$ choose $\Gamma(t) = 2t(\sigma - u/4)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\Gamma(t) = \sigma - u/4 + (t - 2^{-1})\, u$ for $\frac{1}{2} < t \leqslant 1$. Then Γ is in $\mathscr{R}$ and for small s, $\Gamma(s + \frac{3}{4}) = \sigma + su$. So $\hat{P}(\sigma + su) = R(\Gamma^{s + 3/4})^{-1}$ by (3.5). But by (3.4),

$$dR(\Gamma^{s + 3/4})/ds\,|_{s = 0} = -B(\sigma, u)\, R(\Gamma^{3/4}).$$

Therefore $d\hat{P}(\sigma + su)^{-1}/ds\,|_{s=0} = -B(\sigma, u)\, \hat{P}(\sigma)$,which implies (3.3). The uniqueness of $\hat{P}$ follows from the uniqueness of solutions to the differential equation $d\hat{P}(s\sigma)/ds = \hat{P}(s\sigma)\, B(s\sigma, \sigma)$, which is a special case of (3.3), together with $\hat{P}(0) = 1$.

LEMMA 3.7. *Let* ϕ: $R^n \times [0, 1] \to R^n$ *be a map of class* C^∞ *such that* (a) $\phi(x, 0) = 0$ *for all* x *in* R^n, (b) $\phi(x, 1) = x$ *for all* x *in* R^n, *and* (c) $\phi(0, t) = 0$

for all t in $[0, 1]$. (E.g., $\phi(x, t) = tx$ would do.) Put $\phi_i(x, t) = \partial\phi(x, t)/\partial x^i$ and $\dot\phi(x, t) = \partial\phi(x, t)/\partial t$. Define

$$A_i(x) = B(\phi(x, \cdot), \phi_i(x, \cdot)). \tag{3.6}$$

Then $A(x) := \sum_i A_i(x)\, dx^i$ is a $\mathscr{G}$-valued 1-form on R^n of class C^∞. If P denotes parallel translation for A then $\hat P(\sigma) = P(\sigma)$ for all σ in $\mathscr{P}_0$.

Proof. Let σ be in $\mathscr{P}_0$ and of class C^∞. Let $0 < a < 1$ and for each number s in $[0, 1]$ define

$$\begin{aligned} \gamma(s, t) &= \phi(\sigma(s), t/a) \qquad & 0 \leqslant t \leqslant a, \\ &= \sigma(s(1-t)/(1-a)), \qquad & a < t \leqslant 1. \end{aligned} \tag{3.7}$$

For each s $\gamma(s, \cdot)$ is easily seen to be in $\mathscr{P}_0$. Moreover,

$$\begin{aligned} \gamma_s(s, t) &= \sum_i \phi_i(\sigma(s), t/a)\, \dot\sigma_i(s), \qquad & 0 \leqslant t \leqslant a, \\ &= \dot\sigma(s(1-t)/(1-a))(1-t)/(1-a), \qquad & a < t \leqslant 1, \end{aligned} \tag{3.8}$$

so that for each s, $\gamma_s(s, t)$ is continuous in t (e.g., at $t = a$), is zero at the endpoints, C^∞ elsewhere, hence is in $\mathscr{P}_0$. Moreover, $s \to \gamma_s(s, \cdot)$ is continuous into $\mathscr{P}_0$. Now

$$\begin{aligned} \gamma_t(s, t) &= a^{-1}\dot\phi(\sigma(s), t/a), \qquad & 0 \leqslant t < a, \\ &= -\dot\sigma(s(1-t)/(1-a))\, s(1-a)^{-1}, \qquad & a < t \leqslant 1. \end{aligned} \tag{3.9}$$

Note that $\gamma_s(s, t)$ and $\gamma_t(s, t)$ are linearly dependent vectors in R^n for $a < t \leqslant 1$ and further, for $r \leqslant a$, the curve $t \to \gamma(s, t)$ satisfies, for $0 \leqslant t \leqslant 1$,

$$\gamma^r(s, t) = \gamma(s, rt) = \phi(\sigma(s), rt/a) = \phi^{r/a}(\sigma(s), t).$$

Hence

$$\begin{aligned} &B(\gamma(s, \cdot), \gamma_s(s, \cdot)) \\ &= \int_0^1 h(\gamma^r(s, \cdot))\langle \gamma_t(s, r), \gamma_s(s, r)\rangle\, dr \\ &= \int_0^a h(\gamma^r(s, \cdot))\langle a^{-1}\dot\phi(\sigma(s), r/a), \gamma_s(s, r)\rangle\, dr \\ &= \int_0^a h(\phi^{r/a}(\sigma(s), \cdot))\langle \dot\phi(\sigma(s), r/a), \sum_i \phi_i(\sigma(s), r/a)\, \dot\sigma_i(s)\rangle\, a^{-1}\, dr \\ &= \sum_i \int_0^1 h(\phi^u(\sigma(s), \cdot))\langle \dot\phi(\sigma(s), u), \phi_i(\sigma(s), u)\rangle\, du\, \dot\sigma_i(s) \\ &= A(\sigma(s)) \cdot \dot\sigma(s). \end{aligned}$$

Put $g(s) = \hat{P}(\gamma(s, \cdot))$. Then by (3.3),

$$dg(s)/ds = g(s)\, B(\gamma(s, \cdot), \gamma_s(s, \cdot)).$$

Thus

$$dg(s)/ds = g(s)\, A(\sigma(s)) \cdot \dot{\sigma}(s)$$

while $\gamma(0, t) \equiv 0$ so that $g(0) = 1$. Hence $g(s)^{-1}$ satisfies the system (2.3). Therefore $g(1)^{-1} = P(\sigma)$. That is, $\hat{P}(\gamma(1, \cdot)) = P(\sigma)^{-1}$. As is customary, we write $\sigma^{-1}(t) = \sigma(1-t)$. Now

$$\|\gamma(1, \cdot) - \sigma^{-1}\| = \int_a^1 |\dot{\sigma}((1-t)/(1-a))(1-a)^{-1} - \dot{\sigma}(1-t)|\, dt$$

$$+ \int_0^a |\dot{\sigma}(1-t)|\, dt,$$

by (3.7) since $\sigma(1) = 0$. Thus $\|\gamma(1, \cdot) - \sigma^{-1}\|$ converges to zero as a decreases to zero. Since $\hat{P}$ is continuous on $\mathscr{P}_0$ we have $\hat{P}(\gamma(1, \cdot)) \to \hat{P}(\sigma^{-1})$ as $a \downarrow 0$. Hence $\hat{P}(\sigma^{-1}) = P(\sigma)^{-1} = P(\sigma^{-1})$. Since σ is an arbitrary C^∞ closed curve so is σ^{-1}. Thus $\hat{P}(\sigma) = P(\sigma)$ when σ is closed and of class C^∞. But $C^\infty \cap \mathscr{P}_0$ is dense in $\mathscr{P}_0$. The continuity of both $\hat{P}$ and P on $\mathscr{P}_0$ now shows that $\hat{P} = P$ on $\mathscr{P}_0$.

Proof of Theorem 3.4. Define A as in Lemma 3.7 and denote by B^A the associated 1-form on $\mathscr{P}_0$ and by P parallel translation for A. $\hat{P}$ is given by Lemma 3.6. By Lemma 3.7, $\hat{P}(\sigma + su) = P(\sigma + su)$ for σ and u in $\mathscr{P}_0$ and s real. Differentiating with respect to s at $s = 0$ gives $\hat{P}(\sigma)\ B(\sigma, u) = P(\sigma)\ B^A(\sigma, u)$ by (3.3) and (2.16). Therefore $B(\sigma, u) = B^A(\sigma, u)$ for all σ and u in $\mathscr{P}_0$. Thus if L is the lasso form for A then

$$\int_0^1 h(\sigma^r)\langle \dot{\sigma}(r), u(r) \rangle\, dr = \int_0^1 L(\sigma^r)\langle \dot{\sigma}(r), u(r) \rangle\, dr.$$

Fix $0 < s < 1$ and choose a sequence u_k in $\mathscr{P}_0$ which converges to $\delta(r - s)\,\beta$ for some vector β in R^n. Then the preceding equality gives

$$h(\sigma^s)\langle \dot{\sigma}(s), \beta \rangle = L(\sigma^s)\langle \dot{\sigma}(s), \beta \rangle \tag{3.10}$$

provided $\dot{\sigma}$ is continuous at s. Now suppose that γ is in $\mathscr{P}$ and α is in R^n. Let σ be a curve in $\mathscr{P}_0$ such that $\sigma(t) = \gamma(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and is infinitely differentiable on $[\frac{1}{2}, 1]$ with $\lim_{s \downarrow 1/2} \dot{\sigma}(s) = \alpha$. Such a curve σ clearly exists. Then $\sigma^{1/2} = \gamma$. For $s > \frac{1}{2}$ we have (3.10). Take $\lim_{s \downarrow 1/2}$ and use the continuity in s of $h(\sigma^s)$ and $L(\sigma^s)$ to get $h(\gamma)\langle \alpha, \beta \rangle = L(\gamma)\langle \alpha, \beta \rangle$. This proves the theorem.

Proof of Corollary 3.5. The theorem shows that the map $A \to L^A$ maps onto the space of smooth closed path 2-forms on $\mathscr{P}$. It is well known that under a gauge transformation $A \to A^g$ the lasso form transforms by $L^{A^g} = g(0)^{-1} L^A g(0)$. That is, L^A is an operator on the fiber above 0 in the vector bundle $R^m \times \mathbb{C}^N$. Thus $L^{A^g} = L^A$ if $g(\cdot)$ is in the restricted gauge group. It remains only to show that if two smooth $\mathscr{G}$-valued 1-forms A and C on R^n determine the same lasso form L on $\mathscr{P}$ then they are gauge equivalent via a restricted gauge transformation.

Denote by $P^A(\sigma)$ and $P^C(\sigma)$ the parallel translation operators determined by A and C, respectively, for σ in $\mathscr{P}$. By the noncommutative Stokes theorem, Eq. (2.16), $P^A(\gamma) = P^C(\gamma)$ if γ is a closed curve because A and C have the same lasso form. If σ and τ are in $\mathscr{P}$ with $\sigma(1) = \tau(1) = x$ then $\sigma^{-1}\tau$ is a closed curve and therefore

$$P^A(\sigma)^{-1} P^A(\tau) = P^A(\sigma^{-1}\tau) = P^C(\sigma^{-1}\tau) = P^C(\sigma)^{-1} P^C(\tau).$$

Hence $P^C(\sigma) P^A(\sigma)^{-1} = P^C(\tau) P^A(\tau)^{-1}$. Both sides depend, therefore, only on the endpoint x of σ. Put

$$g(x) = P^C(\sigma) P^A(\sigma)^{-1}, \qquad \sigma \text{ in } \mathscr{P}. \tag{3.11}$$

Both factors on the right are smooth functions of σ into G by Theorem 2.2. Therefore g is of class C^∞. Moreover $g(0) = 1$ (use $\sigma \equiv 0$.) Replace σ by σ^s in this equation and differentiate with respect to s, using

$$dP^A(\sigma^s)/ds = -(A(\sigma(s)) \cdot \dot{\sigma}(s)) P^A(\sigma^s),$$

which is Eq. (2.3) (g has a different meaning in (2.3)). Thus $g(\sigma(s)) = P^C(\sigma^s) P^A(\sigma^s)^{-1}$ and so

$$\begin{aligned}\langle dg, \dot{\sigma}(s)\rangle = &-(C(\sigma(s)) \cdot \dot{\sigma}(s)) P^C(\sigma^s) P^A(\sigma^s)^{-1} \\ &+ P^C(\sigma^s) P^A(\sigma^s)^{-1} (A(\sigma(s)) \cdot \dot{\sigma}(s)).\end{aligned}$$

Put $s = 1$ and use (3.11) again to get

$$\langle dg, \dot{\sigma}(1)\rangle = -\langle C(x), \dot{\sigma}(1)\rangle g(x) + g(x)\langle A(x), \dot{\sigma}(1)\rangle.$$

Since σ is any PC^∞ curve in $\mathscr{P}$ terminating at x, $\dot{\sigma}(1)$ is arbitrary. Hence multiplying by $g(x)^{-1}$ on the left gives

$$A(x) = g^{-1}(x) C(x) g(x) + g^{-1}(x) dg(x) \equiv C^g(x).$$

This completes the proof of the corollary. We note that by reversing the order of steps in the last argument one sees easily that $g(\cdot)$ is unique in the sense that the only function g satisfying $A = C^g$ and $g(0) = 1$ is that given by (3.11).

Remark 3.8. Lemma 3.7 used a smooth homotopy ϕ of the identity map on R^n, $\phi(x, 1) = x$, with the constant map $\phi(x, 0) = 0$. We have thus used the contractibility of R^n. Moreover the proof of Lemma 3.6 used the fact that a closed surface generated by a family of closed curves could be shrunk to a point. That is, $\pi_2(R^n) = 0$. In order to extend this Poincaré lemma to vector bundles over a manifold M, in the spirit of Remark 2.15 and Corollary 2.16, these two topological difficulties must be overcome. Since one need only construct the connection form A in local charts local contractibility will suffice, and indeed always holds. It seems likely that a version of Theorem 3.4 and Corollary 3.5 will hold on a manifold if $\pi_2(M) = 0$ and $\pi_1(M) = 0$ and the proof should be similar to the preceding proofs. But the condition $\pi_2(M) = 0$ is not necessary as we know from the example of the Dirac magnetic monopole. The proof of uniqueness of the connection form A in Corollary 3.5 used $\pi_1(R^n) = 0$. Both the existence and uniqueness in case $\pi_2(M) \neq 0$ and $\pi_1(M) \neq 0$ need further exploration.

Remark 3.9. A choice of the homotopy ϕ in Lemma 3.7 determines the gauge in which A is constructed. Thus if $\phi(x, t) = tx$ then one can show that $A(x) \cdot x = 0$ ("radial gauge"). It actually suffices for the purpose of Lemma 3.7 to relax the C^∞ requirement on ϕ. By using a PC^∞ function $\phi(x, \cdot)$ which is linear in x and piecewise linear in t and moves parallel to the jth coordinate axis for $(j-1)/n \leqslant t < j/n$, Eq. (3.6) produces an A in the so-called complete axial gauge. The radial gauge was used by Lichnerowicz [27, Sect. 60] in his proof of the Ambrose–Singer theorem. Although we do not start with a connection in Lemma 3.7 the spirit of this Lemma is very close to that in Sections 56 and 60 of [27]. Moreover, the important equation (60.5) in [27] is a special case of (2.16).

Remark 3.10. The proof of existence of the parallel translation operator $\hat{P}(\sigma)$ for closed curves σ, in Lemma 3.6 depends on the fact that the 1-form B on $\mathscr{P}_0$ has curvature zero, and not on the fact that B arises from a path 2-form h as in (3.1). However, it is the latter that is responsible for the parametrization independence of $\hat{P}$, which follows from $\hat{P} = P$, proved in Lemma 3.7. Here is an example of a 1-form B on $\mathscr{P}_0$ with curvature zero which does not arise from a path 2-form h and for which $\hat{P}$ is not parametrization independent. Let $A(x, t)$ be a $\mathscr{G}$-valued 1-form on R^n for each t in $[0, 1]$ and which is of class C^∞ in x and t. Define parallel translation along a parametrized curve σ in $\mathscr{P}$ by $P(\sigma) = g(1)$, where $g(t)$ is the solution to the initial value problem

$$dg(t)/dt = -(A(\sigma(t), t) \cdot \dot{\sigma}(t))\, g(t), \qquad g(0) = 1.$$

Lemma 2.4 applies to the present situation with only one slight change in the proof. A derivative of A with respect to t appears in (2.11). It is straightforward to verify that one obtains

$$\begin{aligned}\partial^2\psi(s,t)/\partial s\,\partial t|_{s=0} = g(t)^{-1}\,\{&F(t)\langle\sigma_t, u(t)\rangle \\ &+ (\partial_t A)(\sigma(t), t)\cdot u(t)\}\, g(t),\end{aligned}$$

where $F(t)$ is the curvature of the 1-form $A(\cdot, t)$ for fixed t, $g(t) = g(0, t)$, $\sigma(t) = \gamma(0, t)$, and $u(t) = \gamma_s(0, t)$. Therefore if we define, for σ and u in $\mathscr{P}$,

$$\begin{aligned}B(\sigma, u) = \int_0^1 g(t)^{-1}\,\{&F(t)\langle\dot\sigma(t), u(t)\rangle \\ &+ (\partial_t A)(\sigma(t), t)\cdot u(t)\}\, g(t)\, dt \qquad (3.12)\end{aligned}$$

then

$$\partial_u P(\sigma) = P(\sigma)\, B(\sigma, u) - (A(\sigma(1), 1)\cdot u(1))\, P(\sigma) \qquad (3.13)$$

just as in (2.6). Thus the noncommutative Stokes theorem (2.16) continues to hold. This shows that B is a 1-form on $\mathscr{P}_0$ of curvature zero, just as in the proof of Theorem 3.4. However, B does not arise from a path 2-form h, for if it did then one would have, for $u(t) = \phi(t)\,\dot\sigma(t)$, with ϕ in $C_c^\infty(0, 1)$ and $\sigma \in C^\infty \cap \mathscr{P}_0$,

$$B(\sigma, u) = \int_0^1 h(\sigma^t)\langle\dot\sigma(t), \phi(t)\,\dot\sigma(t)\rangle\, dt = 0.$$

But in fact,

$$B(\sigma, u) = \int_0^1 g(t)^{-1}\,\{0 + \phi(t)(\partial_t A)\cdot\dot\sigma(t)\}\, g(t)\, dt.$$

Thus if $(\partial_t A)\cdot\dot\sigma(t) \neq 0$ for some t (which can always be arranged for some path σ if $\partial_t A$ is not identically zero) then $B(\sigma, u) \neq 0$ for a suitable ϕ with support near t. It follows that $P(\sigma)$ is parametrization dependent, for if $\psi(s, t) = t + s\phi(t)$ then $t \to \psi(s, t)$ is a diffeomorphism of $[0, 1]$ for small s, while

$$dP(\sigma \circ (\psi(s, \cdot)))/ds|_{s=0} = P(\sigma)\, B(\sigma, \phi\dot\sigma) \neq 0.$$

Although Theorem 3.4 is a theorem about a path 2-form h, the key condition (3.2) is imposed not directly on h but on the integrated 1-form B given in (3.1). We anticipate that in the quantized theory the 1-form B will play a central role. The next theorem is a version of Poincaré's lemma formulated directly in terms of B.

DEFINITION 3.11. A $\mathscr{G}$-valued linear functional f on $\mathscr{P}_0$ is *strongly continuous* if

$$|f(u)| \leqslant \text{constant } \sup\{|u(t)|: 0 \leqslant t \leqslant 1\}.$$

Note that the $\mathscr{P}_0$ norm (2.1) is slightly stronger than the norm $\sup_t |\sigma(t)|$. Strong continuity of f allows one to extend f to the space $\mathscr{M}_0$ of bounded measurable R^n-valued functions on $(0, 1)$ (which we may identify with bounded measurable R^n-valued functions on $[0, 1]$ which are zero at the endpoints.) For $\mathscr{P}_0$ is dense in the space of continuous R^n-valued functions on $(0, 1)$ vanishing at "∞". By the Riesz representation theorem applied to $(0, 1)$ there is an $(R^n)^* \otimes \mathscr{G}$-valued measure ν such that

$$f(u) = \int_0^1 \langle u(t), d\nu(t) \rangle, \qquad u \text{ in } \mathscr{P}_0. \tag{3.14}$$

The pairing $\langle\ ,\ \rangle$ is between R^n and $(R^n)^*$. (Take bases of R^n and $\mathscr{G}$ and construct $n \cdot \dim \mathscr{G}$ real-valued measures.) The functional of u on the right side of (3.14) clearly extends to $\mathscr{M}_0$ and the extension is unique under the condition of continuity with respect to bounded pointwise convergence of sequences. For example, the map $u \to B(\sigma, u)$ given in (3.1) is clearly strongly continuous. We regard henceforth a strongly continuous linear functional on $\mathscr{P}_0$ to be extended to $\mathscr{M}_0$ in the above manner whenever indicated.

DEFINITION 3.12. A $\mathscr{G}$-valued 1-form $B(\sigma, u)$ on $\mathscr{P}_0$ is *nonanticipating* if for each element σ in $\mathscr{P}_0$ the linear map $u \to B(\sigma, u)$ is strongly continuous, and if for each number s in $(0, 1)$ and each pair σ, τ in $\mathscr{P}_0$ and each u in $\mathscr{M}_0$, $B(\sigma, u) = B(\tau, u)$ whenever $u(t) = 0$ for $t > s$ and $\sigma(t) = \tau(t)$ for $t \leqslant s$.

It is clear from (3.1) that a 1-form B arising from a smooth path 2-form h is nonanticipating.

THEOREM 3.13. *Let $B(\sigma, u)$ be a $\mathscr{G}$-valued* 1*-form on $\mathscr{P}_0$ which is strongly continuous in u for each σ in $\mathscr{P}_0$. Assume that B: $\mathscr{P}_0 \times \mathscr{M}_0$ is of class C^∞ with the* sup *norm on $\mathscr{M}_0$. There exists a $\mathscr{G}$-valued* 1*-form A of class C^∞ on R^n whose integrated lasso form is B* (cf. *Eqs.* (2.4) *and* (2.5)) *if and only if the following three conditions hold*:

(a) *B has curvature zero* (*i.e.*, (3.2) *holds for all σ, u, v in $\mathscr{P}_0$*)

(b) $B(\sigma, u) = 0$ *whenever σ and u are in $\mathscr{P}_0$ and $\dot\sigma(t)$ and $u(t)$ are linearly dependent vectors in R^n for almost every t in* $(0, 1)$.

(c) *B is nonanticipating.*

Proof. The necessity of (a) follows from Theorem 3.4. The necessity of (b) and (c) follows from (3.1). For the converse we need the following lemma.

LEMMA 3.14. *Assume B satisfies* (a) *and* (b). *Define $\hat{P}(\sigma)$ for σ in $\mathscr{P}_0$ as in Lemma* 3.6. *Then $\hat{P}$ is parametrization independent in the sense that if ψ: $[0, 1] \to [0, 1]$ is of class PC^∞, if $\psi(0)=0$, $\psi(1)=1$, and $\psi'(t) \geqslant 0$* a.e., *then*

$$\hat{P}(\sigma \circ \psi) = \hat{P}(\sigma) \tag{3.15}$$

if σ is in $C^\infty \cap \mathscr{P}_0$.

Proof. We note first that Lemma 3.6, ensuring the existence $\hat{P}$, requires only condition (a). Put $\psi_s(t) = s(\psi(t) - t) + t$ for $0 \leqslant s \leqslant 1$. Then $\psi_s(0) = 0$, $\psi_s(1) = 1$, and

$$d\psi_s(t)/dt = s(\psi'(t) - 1) + 1 \geqslant 0$$

a.e. Hence ψ_s maps $[0, 1]$ onto $[0, 1]$ for each s, and therefore $\sigma \circ \psi_s$ is a PC^∞ closed curve in R^n for each s. Moreover, the location of discontinuities of $d\sigma \circ \psi_s/dt$ are independent of s. They are located at the discontinuity points of $\psi'(t)$. The function $s \to \Gamma(s) := \sigma \circ \psi_s$ from $[0, 1]$ into $\mathscr{P}_0$ is of class C^∞ because the derivatives $(d^k\Gamma(s)/ds^k)(t) = \sigma^{(k)}(\psi_s(t))(\psi(t) - t)^k$ are all in $\mathscr{P}_0$. Hence we may applay Eq. (3.3) to get

$$d\hat{P}(\Gamma(s))/ds = \hat{P}(\Gamma(s))\, B(\Gamma(s), d\Gamma(s)/ds).$$

But

$$(d\Gamma(s)/ds)(t) = \dot{\sigma}(\psi_s(t))(\psi(t) - t)$$

while

$$d\Gamma(s)(t)/dt = \dot{\sigma}(\psi_s(t))(s(\psi'(t) - 1) + 1).$$

For each s these two derivatives are therefore linearly dependent vectors in R^n for each t at which $\psi'(t)$ exists. By condition (b) we have then $B(\Gamma(s), d\Gamma(s)/ds) = 0$ for each s. So $d\hat{P}(\Gamma(s))/ds = 0$. Therefore $\hat{P}(\sigma \circ \psi) = \hat{P}(\Gamma(1)) = \hat{P}(\Gamma(0)) = \hat{P}(\sigma)$, which proves (3.15).

Conclusion of the Proof of Theorem 3.13. Choose a function ϕ: $R^n \times [0, 1] \to R^n$ as in Lemma 3.7 (such as $\phi(x, t) = tx$). For each point x in R^n define a path $\tau(x)$ by

$$\begin{aligned} \tau(x)(t) &= \phi(x, 2t), && 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= \phi(x, 2 - 2t), && \tfrac{1}{2} < t \leqslant 1. \end{aligned} \tag{3.16}$$

Then $\tau(x)$ is a curve that retraces itself on the second half of the unit interval. Continuing the notation of Lemma 3.7 we put $\tau_i(x)(t) = \phi_i(x, 2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\tau_i(x)(t) = 0$ for $\frac{1}{2} < t \leqslant 1$. Define

$$A_i(x) = B(\tau(x), \tau_i(x)). \tag{3.17}$$

One sees from (3.16) that every derivative $\partial^k \tau(x)(t)/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$ is continuous in t on $[0, 1]$, vanishes at $t = 0$ and 1 and is C^∞ in t except at $t = \frac{1}{2}$. Hence these x derivatives each define a closed curve in R^n which is in $\mathscr{P}_0$. Since the x derivatives exist uniformly in t the map $x \to \tau(x)$ from R^n to $\mathscr{P}_0$ is of class C^∞. Similarly the maps $x \to \tau_i(x)$, $i = 1, \ldots, n$, are of class C^∞ from R^n to $\mathscr{M}_0$. Since $B(\sigma, u)$ is jointly of class C^∞ on $\mathscr{P}_0 \times \mathscr{M}_0$, it follows that $A_i(x)$ is infinitely differentiable in x.

We may now imitate a part of the proof of Lemma 3.7. Let σ be in $\mathscr{P}_0$ and of class C^∞. Define $\gamma(s, t)$ as in Eq. (3.7) with $a = \frac{1}{2}$. As noted in Lemma 3.7, the function $s \to \gamma(s, \cdot)$ from $[0, 1]$ into $\mathscr{P}_0$ is continuously differentiable. Therefore the function $g(s) := \hat{P}(\gamma(s, \cdot))$ satisfies the differential equation

$$dg(s)/ds = g(s)\, B(\gamma(s, \cdot), \gamma_s(s, \cdot))$$

and has initial condition $g(0) = 1$. Just as in Lemma 3.7, it will follow that $\hat{P}(\gamma(1, \cdot)) = P(\sigma)^{-1}$, where P is a parallel translation for A, as soon as we show that

$$B(\gamma(s, \cdot), \gamma_s(s, \cdot)) = A(\sigma(s)) \cdot \dot{\sigma}(s).$$

To this end choose functions ξ_k in $C^\infty([0, 1])$ such that $\xi_k(t) = 0$ for $0 \leqslant t \leqslant \frac{1}{2}$, $\xi_k(t) = 1$ for $2^{-1} + k^{-1} \leqslant t \leqslant 1$ and $0 \leqslant \xi_k(t) \leqslant 1$ for all t. By (3.8) and (3.9), $\gamma_t(s, t)$ and $\xi_k(t)\, \gamma_s(s, t)$ are linearly dependent vectors in R^n for all t except possibly $t = \frac{1}{2}$, where $\gamma_t(s, t)$ may not exist. Hence by condition (b), $B(\gamma(s, \cdot), \xi_k(\cdot)\, \gamma_s(s, \cdot)) = 0$. Thus

$$B(\gamma(s, \cdot), \gamma_s(s, \cdot)) = B(\gamma(s, \cdot), (1 - \xi_k(\cdot))\, \gamma_s(s, \cdot)),$$

which converges as $k \to \infty$ to $B(\gamma(s, \cdot), \sum_{i=1}^n \tau_i(\sigma(s))\, \dot{\sigma}_i(s))$ by Eq. (3.8) and the continuity of $B(\tau, u)$ with respect to bounded pointwise convergence in the argument u. But the functions $\tau_i(x)(t)$ are zero for $t > \frac{1}{2}$, while $\tau(\sigma(s))(t) = \gamma(s, t)$ for $t \leqslant \frac{1}{2}$. Hence by condition (c),

$$\begin{aligned} B(\gamma(s, \cdot), \gamma_s(s, \cdot)) &= B(\tau(\sigma(s)), \sum_{i=1}^n \tau_i(\sigma(s))\, \dot{\sigma}_i(s)) \\ &= A(\sigma(s)) \cdot \dot{\sigma}(s). \end{aligned}$$

Now since $\sigma(1)=0$, $\gamma(1,t)=0$ for $0\leqslant t\leqslant\frac{1}{2}$ and equals $\sigma(2(1-t))$ for $\frac{1}{2}<t\leqslant 1$ by (3.7). Thus

$$\gamma(1,t)=\sigma(1-\psi(t))=\sigma^{-1}(\psi(t)),$$

where $\psi(t)=0$ for $0\leqslant t\leqslant\frac{1}{2}$ and $\psi(t)=2t-1$ for $\frac{1}{2}<t\leqslant 1$. By Lemma 3.14,

$$\hat{P}(\sigma^{-1})=\hat{P}(\sigma^{-1}\circ\psi)=\hat{P}(\gamma(1,\cdot))=P(\sigma^{-1}).$$

As in the proofs of Lemma 3.7 and Theorem 3.4 we now conclude that $\hat{P}(\sigma)=P(\sigma)$ for all σ in $\mathscr{P}_0$, and by differentiation, using (3.3) and (2.16), $\mathrm{B}(\sigma,u)=B^A(\sigma,u)$ for all σ and u in $\mathscr{P}_0$, where $B^A(\sigma,u)$ is the integrated lasso form (2.5) for A. This proves the theorem.

Remark 3.15. Polyakov [39] used the kernel of B, $L(\sigma)\langle\dot{\sigma}(t),e_i\rangle$, as the basis of an analogy of Yang–Mills theory with string theory. He observed that B satisfies conditions (a) and (b) of Theorem 3.13. In the absence of a representation of B as in (3.1) condition (c) of the theorem is independent of (a) and (b). For example, the map $\sigma, u\to B(\sigma^{-1},u)$ satisfies (a) and (b) but not (c).

Remark 3.16. Giles [21] has shown how to recover gauge potentials from a knowledge of the functions $w(\sigma_1,\ldots,\sigma_k):=\operatorname{trace}(P(\sigma_1)\cdots P(\sigma_k))$ under suitable conditions on given functions $w(\sigma_1,\ldots,\sigma_k)$, $\sigma_j\in\mathscr{P}_0$. His results are in the direction of a Poincaré lemma but are expressed in terms of global data (trace of products of holonomy group elements) rather than infinitesimal data (path 2-forms). Another global version of a Poincaré-like lemma was announced by Kobayashi [47].

4. Endpoint Derivative, Bianci Identity, and Poincaré Lemma

Our objective is to clarify the role of the Bianci identity in our noncommutative Poincaré lemma, Theorem 3.4. We shall see that the Bianci identity can be easily formulated in terms of path 2-forms but is not sufficient to ensure that a path 2-form is a lasso form, unless G is commutative.

We shall reformulate the condition of closedness (cf. Definition 3.2) somewhat more directly in terms of the path 2-form h instead of B. To this end we need a stronger sense of differentiability.

Definition 4.1. A function f on $\mathscr{P}$ with values in a normed vector space V is *strongly differentiable* on $\mathscr{P}$ if it is of class C^1 in the sense of Definition 2.1 and if the derivative $\partial_u f(\sigma)$ in the direction $u\in\mathscr{P}$ is continuous in u in sup norm.

If f is strongly differentiable on $\mathscr{P}$ to a finite dimensional vector space V then for each curve σ in $\mathscr{P}$ there is a vector valued measure ν on $(0, 1]$ to $(R^n)^* \otimes V$ such that

$$\partial_u f(\sigma) = \int_0^1 \langle u(t), d\nu(t)\rangle. \tag{4.1}$$

The pairing $\langle\ ,\ \rangle$ is between $(R^n)^*$ and R^n. Unlike (3.14), ν may now have some mass at $\{1\}$. We put, for α in R^n,

$$(D_\alpha f)(\sigma) = \langle \alpha, \nu(\{1\})\rangle. \tag{4.2}$$

$D_\alpha f$ is called the *endpoint derivative* of f in the direction α.

EXAMPLE 4.2. Let A be a $\mathscr{G}$-valued C^∞ 1-form on R^n. Suppose that ϕ is a function from R^n to operators on $\mathbb{C}^N$ which is also of class C^∞. Define

$$f(\sigma) = P(\sigma)^{-1}\, \phi(\sigma(1))\, P(\sigma) \tag{4.3}$$

for σ in $\mathscr{P}$. Then $D_\alpha f(\sigma)$ exists and gives the covariant derivative of ϕ with respect to A in the sense that

$$(D_\alpha f)(\sigma) = P(\sigma)^{-1}\, \{\partial_\alpha \phi(x) + [\langle A(x), \alpha\rangle, \phi(x)]\}\, P(\sigma), \tag{4.4}$$

where $x = \sigma(1)$. (In particular,

$$(D_{e_j} f)(\sigma) = P(\sigma)^{-1}\, \{\partial_j \phi(x) + [A_j(x), \phi(x)]\}\, P(\sigma).)$$

To see this, use Eq. ((2.6) to compute $\partial_u f(\sigma)$ thus:

$$\begin{aligned}\partial_u f(\sigma) = \{&-B(\sigma, u)\, P(\sigma)^{-1} + P(\sigma)^{-1}\, A(\sigma(1))\cdot u(1)\}\, \phi(\sigma(1))\, P(\sigma)\\ &+ P(\sigma)^{-1}\, (\partial_{u(1)}\phi)(\sigma(1))\, P(\sigma) + P(\sigma)^{-1}\, \phi(\sigma(1))\{P(\sigma)\, B(\sigma, u))\\ &- A(\sigma(1))\cdot u(1)\, P(\sigma)\}\\ =\ & [f(\sigma), B(\sigma, u)] + P(\sigma)^{-1}\, \{(\partial_u(1)\phi)(\sigma(1))\\ &+ [A(\sigma(1))\cdot u(1), \phi(\sigma(1))]\}\, P(\sigma).\end{aligned} \tag{4.5}$$

Since $B(\sigma, u) = \int_0^1 L(\sigma^s)\langle \dot\sigma(s), u(s)\rangle\, ds$ and since $\dot\sigma$ is in $L^1(0, 1)$ while $L(\sigma^s)$ is a continuous function of s, it follows that $\|\partial_u f(\sigma)\|_{\mathscr{G}} \leqslant \text{const.}\ \sup_s |u(s)|$ and that the representing measure ν for $\partial_u f$ has only one atom. It is located at $s = 1$ and is given by the last term in Eq. (4.5). This proves (4.4).

Denote by F the curvature tensor for A and by F_{ij} its components relative to the standard basis $e_1, \ldots, e_n$ of R^n. Take $\phi(x) = F_{ij}(x)$. The function denoted f in this example is now a component of the lasso form itself. Put $L_{ij}(\sigma) = L(\sigma)\langle e_i, e_j\rangle$. The Bianci identity $D_A F = 0$ (exterior covariant derivative of $F=0$) involves A explicitly in its formulation. But expressed in terms of the lasso form $L(\sigma)$, it is linear and independent of A. Thus by (4.4) the equation $D_A F=0$ may be written

$$\sum_{(i,j,k)} (D_i L_{jk})(\sigma)=0 \qquad \text{(Bianci identity)}, \tag{4.6}$$

where $D_i = D_{e_i}$ and the sum is over the three cyclic permutations of the distinct indices i, j, k. More generally, if h is a smooth path 2-form with values in $\mathscr{G}$ for which the endpoint derivative exists, the one can form its exterior derivative which is the path 3-form defined by

$$\begin{aligned}(Dh)(\sigma)\langle\alpha, \beta, \gamma\rangle = {} & D_\alpha h(\sigma)\langle\beta, \gamma\rangle \\ & + D_\gamma h(\sigma)\langle\alpha, \beta\rangle + D_\beta h(\sigma)\langle\gamma, \alpha\rangle.\end{aligned} \tag{4.7}$$

It should be noted that the covariant exterior derivative of h defined by (4.7) is linear in h. Thus if A and C are smooth $\mathscr{G}$-valued connection forms with corresponding lasso forms L^A and L^C then $h(\sigma) \equiv L^A(\sigma) + L^C(\sigma)$ is a strongly differentiable path 2-form and satisfies Bianci's identity $Dh=0$. However, we shall see below that h need not be the lasso form of a connection if G is not commutative, and therefore the Bianci identity is not a sufficient condition for representability as a lasso form.

We note that the Yang–Mills current, made into a path 1-form, $J(\sigma) \equiv P(\sigma)^{-1} (D_A^* F)\, P(\sigma)$, can be computed linearly from the lasso form for A by the equation $J_k(\sigma) = \sum_i D_i L_{ik}(\sigma)$. $J(\sigma)$ is again strongly differentiable. A strongly differentiable path 0-form arises naturally from a smooth section $\psi: R^n \to \mathbb{C}^N$ by putting $\bar\psi(x)\,\psi(x)\,\zeta = (\zeta, \psi(x))_{\mathbb{C}^N}\,\psi(x)$ for ζ in $\mathbb{C}^N$ and defining $(\bar\psi\psi)(\sigma) = P(\sigma)^{-1}\,\bar\psi(\sigma(1))\;\psi(\sigma(1))\;P(\sigma)$. All of these examples, $L_{ij}(\sigma)$, $J_k(\sigma)$, $(\bar\psi\psi)(\sigma)$ act on the fiber, $\mathbb{C}^N$ at 0 when properly interpreted. That is, under gauge transformation by an element g in $C^\infty(R^n; G)$, they all transform by $E \to g(0)^{-1}\, Eg(0)$. Consequently it is conceptually meaningful to form sums such as $\sum_m L_{ij}(\sigma_m)$ even though the endpoints $\sigma_m(1)$ may be different. This fact is the motive for this work as explained in the introduction.

EXAMPLE 4.3. Pick a bounded measurable function v: $[0, 1] \to R^n$ and a smooth $\mathscr{G}$-valued connection form A on R^n. If L is the lasso form for A put $f(\sigma) = B(\sigma, v) \equiv \int_0^1 L(\sigma^s)\langle\dot\sigma(s), v(s)\rangle\, ds$. Then

$$\begin{aligned}\partial_u f(\sigma) = {} & \int_0^1 (\partial_u(L(\sigma^s)))\langle\dot\sigma(s), v(s)\rangle\, ds \\ & + \int_0^1 L(\sigma^s)\langle\dot u(s), v(s)\rangle\, ds\end{aligned}$$

for u in $\mathscr{P}$. The first term is bounded in $\sup_s |u(s)|$ but the second term is not bounded in $\sup_s |u(s)|$ if v is sufficiently irregular, specifically if v is not of bounded variation. In this case f is not strongly differentiable.

EXAMPLE 4.4. Here is an example of lasso type which is strongly differentiable and satisfies Bianci's identity but is parametrization dependent. Consider the situation of Remark 3.10, in which the connection $A(x, t)$ depends on t in $[0, 1]$. Define

$$h(\sigma)\langle\alpha, \beta\rangle = P(\sigma)^{-1} F(\sigma(1), 1)\langle\alpha, \beta\rangle P(\sigma),$$

wherein $F(x, 1)$ is the curvature of $A(x, 1)$. The equation (3.13) yields (cf. Eq. (4.5)).

$$\begin{aligned}(\partial_u h)(\sigma)\langle\alpha, \beta\rangle &= [h(\sigma)\langle\alpha, \beta\rangle, B(\sigma, u)] \\ &\quad + P(\sigma)^{-1} (D_{u(1)}F)(\sigma(1), 1)\langle\alpha, \beta\rangle P(\sigma), \qquad (4.8)\end{aligned}$$

where $D_{u(1)}F$ denotes the covariant derivative of F with respect to the same connection form $A(x, 1)$, for which F is the curvature and B is given by (3.12). So h is strongly differentiable and its endpoint derivative is given by

$$(D_j h)(\sigma)\langle\alpha, \beta\rangle = P(\sigma)^{-1} (D_j F)(\sigma(1), 1)\langle\alpha, \beta\rangle P(\sigma).$$

Thus h satisfies the Bianci identity. However, an argument similar to that in Remark 3.10 shows that $h(\sigma)\langle\alpha, \beta\rangle$ is not an ordinary lasso form of some connection because $h(\sigma)\langle\alpha, \beta\rangle$ is not parametrization independent unless $(\partial_t A_i)(x, t)$ commutes with $F_{jk}(x, 1)$ for all x, y, t, i, j, k.

THEOREM 4.5. *Let h be a $\mathscr{G}$-valued C^∞ path 2-form on $\mathscr{P}$ which is strongly differentiable. Then there exists a $\mathscr{G}$-valued 1-form A of class C^∞ on R^n such that h is the lasso form for A, if and only if h satisfies both*

$$\text{(a)} \quad (\partial_u h)(\sigma) = [h(\sigma), B(\sigma, u)] + D_{u(1)}h(\sigma) \qquad u, \sigma \text{ in } \mathscr{P}, \qquad (4.9)$$

$$\text{(b)} \quad Dh(\sigma) = 0 \qquad \text{(Bianci identity)} \qquad \sigma \in \mathscr{P}, \qquad (4.10)$$

where

$$B(\sigma, u) = \int_0^1 h(\sigma^r) < \dot{\sigma}(r), \qquad u(r) > dr.$$

Proof. If h is the lasso 2-form for a smooth connection form A then (a) is Eq. (2.7) already proven for lasso forms. The Bianci identity (b) holds as noted in Example 4.2.

For the converse, assume (a) and (b) hold. We shall show that $B(\sigma, u)$ has curvature zero. It suffices to prove (3.2) in case σ is in $C^\infty \cap \mathscr{P}_0$, since $\partial_u B(\sigma, v)$ is continuous in σ in $\mathscr{P}_0$ norm by standard arguments.

First note that $d\sigma^s(t)/ds = t\dot\sigma(st)$ which is in $\mathscr{P}$. Thus if we put $u(t) = t\dot\sigma(s_0 t)$ then at $s = s_0$,

$$dh(\sigma^s)/ds = (\partial_u h)(\sigma^s) = [h(\sigma^s), B(\sigma^s, u)] + (D_{u(1)}h)(\sigma^s)$$

by (a). But

$$B(\sigma^s, u) = \int_0^1 h(\sigma^{sr})\langle s\dot\sigma(sr), r\dot\sigma(sr)\rangle\, dr = 0$$

because $h(\sigma)\langle \alpha, \alpha\rangle = 0$. Hence

$$dh(\sigma^s)/ds = (D_{\dot\sigma(s)}h)(\sigma^s). \tag{4.11}$$

Second, note that for any σ and any u in $\mathscr{P}$,

$$\begin{aligned} B(\sigma^s, u^s) &= \int_0^1 h(\sigma^{sr})\langle (\sigma^s)^\cdot\,(r), u^s(r)\rangle\, dr \\ &= \int_0^1 h(\sigma^{sr})\langle s\dot\sigma(sr), u(sr)\rangle\, dr. \end{aligned}$$

So putting $sr = t$ gives

$$B(\sigma^s, u^s) = \int_0^s h(\sigma^t)\langle \dot\sigma(t), u(t)\rangle\, dt. \tag{4.12}$$

For σ in $C^\infty \cap \mathscr{P}_0$ and u and v in $\mathscr{P}_0$,

$$\begin{aligned} &\partial_u B(\sigma, v) - \partial_v B(\sigma, u) \\ &\quad = \int_0^1 ((\partial_u s)\, h)(\sigma^s)\langle \dot\sigma(s), v(s)\rangle\, ds \\ &\qquad - \int_0^1 ((\partial_v s) h)(\sigma^s)\langle \dot\sigma(s), u(s)\rangle\, ds \\ &\qquad + \int_0^1 \{h(\sigma^s)\langle \dot u(s), v(s)\rangle - h(\sigma^s)\langle \dot v(s), u(s)\rangle\}\, ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \{[h(\sigma^s)\langle\dot\sigma(s), v(s)\rangle, B(\sigma^s, u^s)] \\
&\quad + (D_{u(s)}h)(\sigma^s)\langle\dot\sigma(s), v(s)\rangle\}\, ds \\
&\quad - \int_0^1 \{[h(\sigma^s)\langle\dot\sigma(s), u(s)\rangle, B(\sigma^s, v^s)] \\
&\quad + (D_{v(s)}h)(\sigma^s)\langle\dot\sigma(s), u(s)\rangle\}\, ds \\
&\quad - \int_0^1 (dh(\sigma^s)/ds)\langle u(s), v(s)\rangle\, ds
\end{aligned}
$$

by (a) and by an integration by parts. Use (4.11) on the last term and observe that

$$
\begin{aligned}
&(D_{u(s)}h)(\sigma^s)\langle\dot\sigma(s), v(s)\rangle + (D_{v(s)}h)(\sigma^s)\langle u(s), \dot\sigma(s)\rangle \\
&\quad + (D_{\dot\sigma(s)}h)(\sigma^s)\langle v(s), u(s)\rangle = 0
\end{aligned}
$$

by the Bianci identity (b). Thus by (4.12),

$$
\begin{aligned}
&\partial_u B(\sigma, v) - \partial_v B(\sigma, u) \\
&= \int_0^1 \left[h(\sigma^s)\langle\dot\sigma(s), v(s)\rangle, \int_0^s h(\sigma^t)\langle\dot\sigma(t), u(t)\rangle \right] dt\, ds \\
&\quad + \int_0^1 \left[\int_0^s h(\sigma^t)\langle\dot\sigma(t), v(t)\rangle, h(\sigma^s)\langle\dot\sigma(s), u(s)\rangle \right] dt\, ds.
\end{aligned}
$$

Interchange the names of the variables t and s in the second term and then reverse the order of integration to get

$$
\int_0^1 \left[h(\sigma^s)\langle\dot\sigma(s), v(s)\rangle, \int_s^1 h(\sigma^t)\langle\dot\sigma(t), u(t)\rangle \right] dt\, ds
$$

for the second term. Adding the two terms gives $[B(\sigma, v), B(\sigma, u)]$ on the right. This proves that (3.2) holds.

Remark 4.6. If G is commutative then Theorem 4.5 reduces to the classical Poincaré lemma for 2-forms. For the commutator on the right of (4.9) is now zero so that (4.9) reads $\partial_u h(\sigma) = D_{u(1)} h(\sigma)$. This equation is clearly satisfied if $h(\sigma)$ depends only on the endpoint of σ. Conversely, if $u(1) = 0$ then (4.9) implies $\partial_u h(\sigma) = 0$, which shows that $h(\sigma)$ depends only on the endpoint of σ. Thus (4.9) is equivalent to the assertion that $h(\sigma) = F(\sigma(1))$ for some 2-form F on R^n. Equation (4.10) is now equivalent to the statement that $dF = 0$. Thus if G is commutative then (4.9) and (4.10) together are equivalent to the usual hypothesis of the classical Poincaré Lemma.

Remark 4.7. If G is not commutative then the Bianci identity (4.10) is not sufficient to ensure that a path 2-form h is the lasso form of some $\mathscr{G}$-valued connection. For (4.10) is linear while (3.2) shows that if h is a lasso form then sh is not a lasso form for any real number s except $s=1$. This argument is valid if for some σ, u, v in $\mathscr{P}_0$ the commutator in (3.2) is not zero. That this can occur for some connections follows from the fact that $\{B(\sigma, u)\colon \sigma, u \in \mathscr{P}_0\}$ generates the Lie algebra of the holonomy group (cf. Corollary 2.11) which can be noncommutative.

EXAMPLE 4.8. Here is an explicit example of two lasso forms L^A and L^C whose sum is not a lasso form (nor is any nontrivial linear combination of them a lasso form.) Take $R^n = R^3$, $G = SU(2)$, $A(x) = Sa(x)$, $C(x) = Tc(x)$, where S and T are any noncommuting elements in $su(2)$ and a and c are real-valued C^∞ 1-forms on R^3. Then $F^A = S\,da$ and F^A commutes with $P^A(\sigma)$ for all σ. Hence $L^A(\sigma) = F^A(\sigma(1))$ and depends only on the endpoint of σ. Similarly $L^C(\sigma) = F^C(\sigma(1))$. Since L^A and L^C both satisfy Eq. (4.9), their sum $h = L^A + L^C$ will fail to satisfy it if the cross terms in the commutator do not add up to zero. Put $L_{ij}(\sigma) = L(\sigma)\langle e_i, e_j\rangle$, where $e_1,\ldots, e_3$ is the standard basis of R^3. It suffices to show that for a suitable choice of a and c and C^∞ curves σ and u,

$$[L^A_{23}(\sigma), B^C(\sigma, u)] + [L^C_{23}(\sigma), B^A(\sigma, u)] \neq 0.$$

We shall choose σ to lie along the x_1 axis and u to point in the e_2 direction. By choosing u to be supported in a small neighborhood of some point s in $[0, 1]$ it suffices to show that

$$[L^A_{23}(\sigma), L^C_{12}(\sigma^s)\,\dot{\sigma}_1(s)] + [L^C_{23}(\sigma), L^A_{12}(\sigma^s)\,\dot{\sigma}_1(s)] \neq 0$$

for some $s > 0$. Choose $a_2 = a_3 = 0$ and a_1 in $C^\infty(R^3)$. Then $L^A_{23}(\sigma) = 0$ and $L^A_{12}(\sigma^s) = -S(\partial_2 a_1)(\sigma(s))$. Choose $c_1 = c_3 = 0$ and c_2 in $C^\infty(R^3)$. Then $L^C_{23}(\sigma) = -T(\partial_3 c_2)(\sigma(1))$. Thus the first commutator is zero while the second is $[T, S](\partial_3 c_2)(\sigma(1))(\partial_2 a_1)(\sigma(s))\,\dot{\sigma}_1(s)$. Clearly this is nonzero for a suitable choice of $c_2(x)$, $a_1(x)$, σ_1, and s.

Remark 4.9. The derivative of a path function f can be defined at an interior point by Eq. (4.1) as well as at the endpoint. The useful notion of interior derivative however is different from that of endpoint derivative. Assume that the measure v in (4.1) is absolutely continuous when restricted to the open interval $(0, 1)$. Then we may write

$$\partial_u f(\sigma) = \int_0^1 \langle u(t), K_\sigma(t)\rangle\, dt + \langle u(1), v\{1\}\rangle, \tag{4.13}$$

with K_σ an integrable function on $(0, 1)$ to $(R^n)^* \otimes V$ if f is V-valued. The component of $K_\sigma(t)$ in the ith coordinate direction will be written

$$\partial f(\sigma)/\partial\sigma^i(t) := \langle e_i, K_\sigma(t)\rangle, \qquad t<1. \tag{4.14}$$

EXAMPLE 4.10. If $L(\sigma)$ is the lasso form for some smooth connection form A on R^n then by Eq. (2.7),

$$\partial_u L_{jk}(\sigma) = \int_0^1 [L_{jk}(\sigma), L(\sigma^r)\langle\dot\sigma(r), u(r)\rangle]\, dr$$
$$+ D_{u(1)} L_{jk}(\sigma).$$

Therefore, for σ in $\mathscr{P}$,

$$\partial L_{jk}(\sigma)/\partial\sigma^i(t) = [L_{jk}(\sigma), L(\sigma^t)\langle\dot\sigma(t), e_i\rangle], \qquad t<1.$$

The right side exists for every t at which $\dot\sigma(t)$ exists.

The next theorem is the fourth, and least integrated form of the noncommutative Poincaré lemma.

THEOREM 4.11. *Let h be a $\mathscr{G}$-valued path 2-form on $\mathscr{P}$. There exists a smooth $\mathscr{G}$-valued 1-form A on R^n such that h is the lasso form for A if and only if h is of class C^∞ on $\mathscr{P}$, is strongly differentiable with representing measures* (4.1) *for $\partial_u h_{jk}(\sigma)$ which are absolutely continuous on* $(0, 1)$, *and the following both hold*:

$$\partial h_{jk}(\sigma)/\partial\sigma^i(t) = [h_{jk}(\sigma), h(\sigma^t)\langle\dot\sigma(t), e_i\rangle] \qquad \text{for a.e. } t<1, \tag{4.15}$$

$$Dh(\sigma) = 0. \tag{4.16}$$

Proof. The necessity of these conditions follows from Theorem 4.5 and Example 4.10. Conversely, under the stated conditions on h, we may write for u in $\mathscr{P}$,

$$\partial_u h_{jk}(\sigma) = \int_{(0,1]} \langle u(t), dv(t)\rangle$$
$$= \int_{(0,1)} \langle u(t), dv(t)\rangle + \langle u(1), v(\{1\})\rangle$$
$$= \sum_{i=1}^n \int_0^1 [h_{jk}(\sigma), h(\sigma^t)\langle\dot\sigma(t), e_i\rangle]\, u_i(t)\, dt$$
$$+ D_{u(1)} h_{jk}(\sigma) = [h_{jk}(\sigma), B(\sigma, u)] + D_{u(1)} h_{jk}(\sigma).$$

Therefore Eqs. (4.9) and (4.10) both hold. We may apply Theorem 4.5 to complete the proof.

COROLLARY 4.12. *Theorem* 4.5 *remains valid if Eq.* (4.9) *is replaced by*

$$(\partial_u h)(\sigma) + [B(\sigma, u), h(\sigma)] = 0, \qquad \sigma \text{ in } \mathscr{P},\ u \text{ in } \mathscr{P}_0. \tag{4.17}$$

Proof. Equation (4.17) shows that the absolute continuity hypothesis of Theorem 4.11 holds as pointed out in Example 4.10. Moreover (4.17) implies (4.15).

Remark 4.13. By rearranging the hypotheses in Theorem 4.5 or Corollary 4.12 one can dispense with the technical assumption of strong differentiability. Thus if one assumes that h is continuously differentiable in the sense of Definition 2.1 and that (4.17) holds then it follows easily (write $u(t) = (u(t) - tu(1)) + tu(1)$) that h is strongly differentiable. Equation (4.10) then makes sense.

Remark 4.14. There are more geometrical ways to define endpoint derivative and interior point derivative of a function f on $\mathscr{P}$ provided $f(\sigma)$ is independent of the parametrization of σ. If f is parametrization independent it is natural to define the endpoint derivative $D_i f(\sigma)$ as $df(\sigma_s)/ds|_{s=0}$, where σ_s is the curve σ extended beyond its endpoint a distance s parallel to the ith coordinate axis and parametrized in any way on $[0, 1]$. This definition was used by Birula [11] and coincides with our $D_i f(\sigma)$ in case f is parametrization independent. But it is not well defined if f is not parametrization independent. Similarly, if f is parametrization independent the derivative $\partial f(\sigma)/\partial \sigma^i(t)$ at an interior point may be defined by inserting a small rectangular hairpin in σ at $\sigma(t)$ and lying in the plane spanned by $\dot{\sigma}(t)$ and e_i. One then takes the limit of a difference quotient with the area "enclosed" by the hairpin in the denominator. If f is a function of parallel translation operators this is a geometrically suggestive definition. This was used by Mandelstam and Migdal [30, 31]. Again, parametrization independence of f seems to be required in order for this definition to be natural.

5. THE EQUATIONS OF QUANTUM CHROMODYNAMICS WITHOUT POTENTIALS

The standard argument showing the equivalence of Maxwell's equations with the potential equation is as follows. Maxwell's equations on four dimensional Minkowski space R^4 may be written

$$dF = 0 \tag{5.1}$$

$$d^*F = j. \tag{5.2}$$

Here F is a real-valued 2-form on R^4, j is a given 1-form on R^4, and d^* is the adjoint of the exterior derivative operator d, computed with respect to the Lorentz metric $(1, -1, -1, -1)$. By Eq. (5.1) and the classical Poincaré lemma, there exists a 1-form A (a potential for F) such that $F = dA$. For such a 1-form equation (5.2) reads

$$d^* \, dA = j \qquad \text{(potential equation).} \tag{5.3}$$

A is not unique, but if A' is another 1-form with $dA' = F$ then $d(A' - A) = 0$. So applying the classical Poincaré lemma a second time we are assured of the existence of a real-valued function λ such that $A' = A + d\lambda$. We may take $\lambda(0) = 0$. The Yang–Mills equation is a non-abelian generalization of the potential equation (5.3). It is the purpose of this section to describe non-Abelian generalizations of the Maxwell equations (5.1), (5.2) and to show their equivalence to the Yang–Mills equation in precisely the same spirit of equivalence as given above. The first use of the classical Poincaré lemma in the preceding argument will be replaced by our noncommutative Poincaré lemma, Theorem 4.5. The second use will be replaced by Corollary 3.5.

The key points in the preceding argument are as follows. For simplicity and precision we use the C^∞ category throughout. Let j be a smooth 1-form on R^4,

(a) The map $A \to F^A \equiv dA$ maps the smooth solutions of Eq. (5.3) onto the smooth solutions of (5.1), (5.2).

(b) The map $A \to F^A$ is one to one as a map from the restricted gauge orbit space. That is, if A and A' are both of class C^∞ and $F^A = F^{A'}$ then A is equivalent to A' by a unique restricted gauge transformation of class C^∞.

Let G be a Lie subgroup of $U(N)$ and $\mathscr{G}$ its Lie algebra. G may be regarded as the image of some fundamental gauge group under an N-dimensional unitary representation. As in the preceding sections we regard $\mathscr{G}$ as a subset of the set of skew Hermitian operators on $\mathbb{C}^N$. Let $j = (j_\mu(x))$ be a smooth $\mathscr{G}$-valued 1-form on R^4. The Yang–Mills equation for a $\mathscr{G}$-valued 1-form A on R^4 with source j is

$$D_A^*(dA + A \wedge A) = j. \tag{5.4}$$

As is well known [25] the field strength,

$$F = F^A := dA + A \wedge A, \tag{5.5}$$

associated to A, satisfies the Bianci identity

$$D_A F = 0 \tag{5.6}$$

and by (5.4) one also has

$$D_A^* F = j. \tag{5.7}$$

Because of the similarity of (5.6) and (5.7) to (5.1) and (5.2) (to which (5.6) and (5.7) reduce in case $G = U(1)$), it is very compelling to regard the 2-form F given by (5.5) as the non-Abelian analog of the electromagnetic field strength F (to which (5.5) reduces in case $G = U(1)$.) However, the analogy is a poor one in some significant respects. First of all the equations (5.6) and (5.7), unlike (5.1) and (5.2), are not intrinsic equations for F, since one needs to know the potential A just to formulate them. And once one knows A then F is given by (5.5). In short, Eq. (5.7) is really just the Yang–Mills equation (5.4) for the unknown A in a mildly disguised notation. Second, item (b) above (which is an issue independent of the Yang–Mills equation) breaks down if G is non-Abelian.

The failure of item (b) has been discussed extensively in the physics literature beginning with an example of Wu and Yang [46]. For a very perspicuous discussion of this problem (the so called "gauge field copy" problem), its current status, and an extensive bibliography see the recent paper [33] by Mostow and Shnider.

It is proposed here to use the lasso form

$$L^A(\sigma) = P^A(\sigma)^{-1} F^A(\sigma(1)) P^A(\sigma)$$

as the nonabelian analog of the electromagnetic field strength instead of F^A itself. Indeed $L^A(\sigma)$ reduces to the field strength F if $G = U(1)$, and moreover we have available our Poincaré lemma which distinguishes lasso forms L^A from among general path 2-forms just as (5.1) distinguishes exact 2-forms $F = dA$ from among general 2-forms.

For an arbitrary continuous path 2-form L on $\mathscr{P} := \mathscr{P}(R^4)$ we write, for σ and u in $\mathscr{P}$,

$$B(\sigma, u) = B^L(\sigma, u) = \int_0^1 L(\sigma^r)\langle \dot\sigma(r), u(r)\rangle \, dr.$$

THEOREM 5.1 (Maxwell form of Yang–Mills equation). *Let J be a smooth path 1-form on $\mathscr{P}$:*

(a) *If L is a strongly differentiable and smooth $\mathscr{G}$-valued path 2-form on $\mathscr{P}$ satisfying*

$$\partial_u L(\sigma) + [B(\sigma, u), L(\sigma)] = 0 \tag{5.8}$$

for σ in $\mathscr{P}$ and u in $\mathscr{P}_0$,

$$DL(\sigma) = 0 \qquad \sigma \in \mathscr{P}, \tag{5.9}$$

$$D^*L(\sigma) = J(\sigma) \qquad \sigma \in \mathscr{P}, \tag{5.10}$$

then there exists a smooth $\mathscr{G}$-valued 1*-form A on R^4 and a smooth $\mathscr{G}$-valued* 1*-form j on R^4 such that*

$$L = L^A \tag{5.11}$$

and

$$J_i(\sigma) = P^A(\sigma)^{-1}\, j_i(\sigma(1))\, P^A(\sigma), \qquad i = 0, 1, 2, 3. \tag{5.12}$$

A and j satisfy the Yang–Mills equation (5.4). *Conversely if A and j are smooth* 1*-forms satisfying* (5.4) *then L^A and J satisfy* (5.8), (5.9), *and* (5.10).

(b) *If A' is another smooth $\mathscr{G}$-valued* 1*-form on R^4 with $L^{A'} = L^A$ then there exists a unique function g: $R^4 \to G$ of class C^∞ with $g(0) = 1$ such that*

$$A'(x) = g(x)^{-1}\, A(x)\, g(x) + g(x)^{-1}\, dg(x). \tag{5.13}$$

Moreover if (5.12) *is satisfied by A', j' (i.e., $J(\sigma) = P^{A'}(\sigma)^{-1}\, j'(\sigma(1))\, P^{A'}(\sigma)$) then*

$$j'(x) = g(x)^{-1}\, j(x)\, g(x). \tag{5.14}$$

Remark 5.2. Parts (a) and (b) of the theorem are precise non-Abelian analogs of items (a) and (b) stated above in the discussion of Maxwell's equations. It should be emphasized that the endpoint derivative operator D appearing in (5.8)–(5.10) does not depend on any potential but rather is an intrinsic operator on path forms (cf. Sect. 4.) Equations (5.9) and (5.10) were first observed by Birula [11] for non-Abelian gauge fields. Equation (5.8) was first observed by Mandelstam [30]. The definitions of derivative used in both of these works tacitly assume that the path function being differentiated is parametrization independent, which it is in their case. See Remark 4.14.

Proof. By (5.8), (5.9), and Corollary 4.12 there exists a $\mathscr{G}$-valued smooth 1-form A on R^4 such that (5.11) holds. Denote by D_i endpoint differentiation in the ith coordinate direction. Since $L(\sigma) = L^A(\sigma)$ we have, by Eq. (4.4),

$$D_i L(\sigma) = P^A(\sigma)^{-1}((D_A)_i F^A)(\sigma(1))\, P^A(\sigma). \tag{5.15}$$

Hence by (5.10),

$$J(\sigma) = P^A(\sigma)^{-1}\, ((D_A)^*\, F^A)(\sigma(1))\, P^A(\sigma).$$

Let $j(x) = (D_A^* F^A)(x)$. Then (5.11) and (5.4) hold. Conversely, if A and j are smooth $\mathscr{G}$-valued 1-forms on R^4 satisfying the Yang–Mills equation

(5.4) then $L \equiv L^A$ satisfies (5.8) and (5.9), by Corollary 4.12. Equation (5.10) is satisfied by L^A if J is given by (5.12) in view of (5.15).

For part (b) note that Corollary 3.5 asserts that A' and A are gauge equivalent by a restricted smooth function g as required for (5.13). Moreover g is uniquely determined, as noted at the end of the proof of Corollary 3.5, and satisfies (see Eq. (3.11))

$$g(x)\, P^{A'}(\sigma) = P^A(\sigma), \qquad \sigma(1) = x,\ \sigma \in \mathscr{P}. \tag{5.16}$$

Substituting this for $P^A(\sigma)$ in (5.12) and cancelling $P^{A'}(\sigma)$ on the left and right yields (5.14).

The elimination of the gauge potential from the matter field equations can be accomplished in a similar way. We illustrate the procedure with the coupled Yang–Mills–Dirac equation. The Dirac equation for a particle of mass m in the presence of a $\mathscr{G}$-valued gauge potential A is

$$\sum_{\mu=0}^{3} \gamma^\mu(\partial_\mu + A_\mu)\, \psi + \mathrm{im}\, \psi = 0. \tag{5.17}$$

Here ψ is to be regarded as a section of a vector bundle over R^4 with fiber $\mathbb{C}^N \otimes \mathbb{C}^4$. The Dirac matrices γ^μ act, as usual, on the factor $\mathbb{C}^4$ and satisfy $\gamma_0^2 = -\gamma_i^2 = 1$, $i = 1, 2, 3$; $\gamma_0^* = \gamma_0$; $\gamma_i^* = -\gamma_i$, $i = 1, 2, 3$; while A_μ acts on $\mathbb{C}^N$. Of course $\partial_\mu = \partial/\partial x^\mu$, $\mu = 0, \ldots, 3$. The Dirac equation (5.17) is coupled to the Yang–Mills equation (5.4) by specifying j as a quadratic functional of ψ, which we shall do later. Let us first discuss the key point in eliminating the potential from (5.17). Given a $\mathbb{C}^N \otimes \mathbb{C}^4$ valued function ψ on R^4 and a potential A define a path function Ψ by

$$\Psi(\sigma) = P^A(\sigma)^{-1}\, \psi(\sigma(1)), \qquad \sigma \in \mathscr{P}. \tag{5.18}$$

By Eq. (2.6) we have, for u in $\mathscr{P}$,

$$\begin{aligned} \partial_u \Psi(\sigma) = {} & (-B(\sigma, u)\, P^A(\sigma)^{-1} + P^A(\sigma)^{-1}\, A(\sigma(1)) \cdot u(1))\, \psi(\sigma(1)) \\ & + P^A(\sigma)^{-1}\, (\partial_{u(1)} \psi)(\sigma(1)). \end{aligned} \tag{5.19}$$

Therefore from the definition of endpoint derivative (Sect. 4) in the ith coordinate direction, we have

$$D_i \Psi(\sigma) = P^A(\sigma)^{-1}\, (\partial_i + A_i(x))\, \psi(x), \qquad x = \sigma(1). \tag{5.20}$$

If ψ satisfies (5.17) then the system of equations

$$\partial_u \Psi(\sigma) + B(\sigma, u)\, \Psi(\sigma) = 0 \qquad u \in \mathscr{P}_0, \tag{5.21}$$

$$\sum_{\mu=0}^{3} \gamma^\mu D_\mu \Psi(\sigma) + \mathrm{im}\, \Psi(\sigma) = 0 \tag{5.22}$$

now follows, the first from (5.19) with $u(1)=0$, the second from (5.17) upon multiplying by $P^A(\sigma)^{-1}$ on the left and using (5.20).

The next theorem shows that the system (5.8), (5.9), (5.10), (5.21), (5.22) of potential independent equations is equivalent to the Yang–Mills–Dirac equations (5.4), (5.17). Specifically, the map A, $\psi \to L^A$, Ψ maps the smooth solutions of the latter onto the smooth solutions of the former, and fails to be one to one exactly in accordance with the restricted gauge group. Thus, given the chart of the instantaneous observer at the origin of space–time, it is the solutions of the former system which are in one to one correspondence with the physical states of the classical QCD field.

The current j obtained from the matter field ψ which is to be inserted into (5.4) is defined as follows. Write

$$\psi(x)=\sum_{\alpha=1}^{4} \psi_\alpha(x)\otimes e_\alpha,$$

where $e_1,\ldots, e_4$ is the standard basis of $\mathbb{C}^4$ and each $\psi_\alpha(x)$ is in $\mathbb{C}^N$. For vectors u and v in $\mathbb{C}^N$ let $u\otimes v^*$ denote the operator on $\mathbb{C}^N$ defined by $u\otimes v^*\xi=(v,\xi)_{\mathbb{C}^N}u$ (the inner product is taken to be linear on the right.) Put

$$(\bar{\psi}\gamma_\mu\psi)(x)=\sum_{\alpha=1}^{4} (\gamma_0\gamma_\mu\psi)_\alpha(x)\otimes\psi_\alpha(x)^*. \tag{5.23}$$

This is an operator on $\mathbb{C}^N$ for each ψ and each x and each μ in $\{0,\ldots, 3\}$. Since $\gamma_0\gamma_\mu$ is Hermitian, the operator $(\bar{\psi}\gamma_\mu\psi)(x)$ is a Hermitian operator on $\mathbb{C}^N$, as one sees by using $(u\otimes v^*)^*=v\otimes u^*$ and expanding:

$$(\gamma_0\gamma_\mu\psi)_\alpha(x)=\sum_\beta (\gamma_0\gamma_\mu)_{\alpha\beta}\,\psi_\beta(x).$$

Therefore $i(\bar{\psi}\gamma_\mu\psi)(x)$ is skew-Hermitian and hence is in $u(N)$, the Lie algebra of $U(N)$; $u(N)$ is a real inner product space in the trace inner product, trace (A^*B), and $\mathscr{G}$ is a subspace of $u(N)$. Let

$$\eta\colon u(N)\to\mathscr{G} \tag{5.24}$$

be the orthogonal projection. We then define

$$j_\mu(x)=\eta i(\bar{\psi}\gamma_\mu\psi)(x), \qquad \mu=0,\ldots, 3. \tag{5.25}$$

LEMMA 5.3. *If g is an element of G then the current j_μ^g arising from $g\psi(x)$ is related to the current j_μ arising from ψ by*

$$j_\mu^g(x)=g j_\mu(x)\, g^{-1}. \tag{5.26}$$

Proof. If u and v are in $\mathbb{C}^N$ and ξ is in $\mathbb{C}^N$ then

$$(gu)\otimes(gv)^*\,\xi=(gv,\xi)\,gu=(v,g^{-1}\xi)\,gu=g\,u\otimes v^*g^{-1}\xi.$$

By linearity we therefore have $(\overline{g\psi}\gamma_\mu g\psi)(x)=g(\bar{\psi}\gamma_\mu\psi)(x)\,g^{-1}$. ($g$ commutes with the γ matrices because they act on different factors.) The map $B\to gBg^{-1}$ is a unitary operator on $u(N)$ in the trace inner product. Since it leaves the range of the projection η invariant, it commutes with η. Thus

$$\begin{aligned} j^g_\mu(x)&=\eta i(\overline{g\psi}\gamma_\mu\,g\psi)(x)=\eta g i(\bar{\psi}\gamma_\mu\psi)(x)\,g^{-1}\\ &=gj_\mu(x)\,g^{-1}. \end{aligned}$$

Given a path function $\Psi\colon\mathscr{P}\to\mathbb{C}^N\otimes\mathbb{C}^4$ we may define its current the same way as above:

$$J_\mu(\sigma)=\eta i(\bar{\Psi}\gamma_\mu\Psi)(\sigma),\qquad \sigma\in\mathscr{P}. \tag{5.27}$$

Of course $J_\mu(\sigma)$ transforms as in (5.26) under the map

$$\Psi(\sigma)\to g\Psi(\sigma).$$

THEOREM 5.4. (a) (Equivalence of systems). Let L be a $\mathscr{G}$-valued function on $\mathscr{P}$ and Ψ a $(\mathbb{C}^N\otimes\mathbb{C}^4)$-valued function on $\mathscr{P}$, both of class C^∞, which satisfy the system (5.8)–(5.10), (5.21), (5.22), (5.27). Then there exists a $\mathscr{G}$-valued 1-form A on R^4 and a $(\mathbb{C}^N\otimes\mathbb{C}^4)$-valued function ψ on R^4, both of class C^∞, which satisfy the Yang–Mills–Dirac equations (5.4), (5.17), (5.25), and such that $L=L^A$ while Ψ is given by (5.18).

Conversely, if A and ψ are smooth solutions to the system (5.4), (5.17), (5.25) then $L:=L^A$ and Ψ, given by (5.18), satisfy (5.8)–(5.10), (5.21), (5.22), (5.27) and are of class C^∞ on $\mathscr{P}$.

(b) (Uniqueness up to restricted gauge equivalence). If A, ψ and A', ψ' are two pairs of smooth functions such that $L^A=L^{A'}$ and

$$P^A(\sigma)^{-1}\,\psi(\sigma(1))=P^{A'}(\sigma)^{-1}\,\psi'(\sigma(1))$$

for all σ in $\mathscr{P}$ then there is a unique smooth function $g(\cdot)$ in the restricted local gauge group such that (5.13) holds and such that

$$\psi'(x)=g(x)^{-1}\,\psi(x),\qquad x\in R^4. \tag{5.28}$$

Remark 5.5. Part (b) of this theorem has nothing to do with the question of whether the given data A, ψ satisfies any differential equations. However, if A, ψ satisfies the Yang–Mills–Dirac equations so does A', ψ' as we see by using part (b) and the known gauge invariance of this system.

Proof of Theorem 5.4. (a) Given smooth solutions L and Ψ as indicated, the existence of a smooth function A such that $L = L^A$ follows from (5.8), (5.9), and Corollary 4.12. By Eqs. (2.16) and (5.21) we have, for u in $\mathscr{P}_0$,

$$\partial_u(P^A(\sigma)\,\Psi(\sigma)) = P^A(\sigma)\,B(\sigma, u)\,\Psi(\sigma) + P^A(\sigma)\,\partial_u\Psi(\sigma) = 0. \qquad (5.29)$$

Consequently $P^A(\sigma)\,\Psi(\sigma)$ depends only on the endpoint of σ, for if $\sigma(1) = \tau(1)$, put $u(t) = \tau(t) - \sigma(t)$ and conclude from (5.29) that $P^A(\sigma + su)$ $\Psi(\sigma + su)$ is constant in s. Put $s = 0$ or 1. Define $\psi(x) = P^A(\sigma)\,\Psi(\sigma)$ in case $x = \sigma(1)$. By using the straight line segment $\sigma(t) = tx$, $0 \leqslant t \leqslant 1$, we see that ψ is a smooth function on R^4. Clearly (5.18) is satisfied. Inserting Eq. (5.20) into (5.22) and multiplying by $P^A(\sigma)$ on the left, we see that ψ satisfies the Dirac equation (5.17). Combining (5.18), (5.26) with $g = P^A(\sigma)^{-1}$, and (5.27) yields

$$J_\mu(\sigma) = P^A(\sigma)^{-1}\,j_\mu(x)\,P^A(\sigma),$$

where $x = \sigma(1)$ and j_μ is given by (5.25). Thus by Theorem 5.1, A satisfies the Yang–Mills equation (5.4) with source j given correctly by (5.25).

Conversely if A and ψ are smooth solutions of (5.4), (5.17), and (5.25) then $L := L^A$ satisfies (5.8), (5.9), and (5.10) with J given in terms of j by (5.12) as we already know from Theorem 5.1. Moreover J is also correctly given in terms of Ψ by (5.27) in view of (5.18) and (5.26). We already saw that (5.21) and (5.22) hold when they were introduced. This concludes the proof of part (a).

For part (b) the argument leading to a unique smooth function $g(\cdot)$ satisfying (5.13) is the same as in Theorem 5.1. Moreover (5.28) follows from (5.16) and the hypothesis

$$P^A(\sigma)^{-1}\,\psi(\sigma(1)) = P^{A'}(\sigma)^{-1}\,\psi'(\sigma(1)).$$

Remark 5.6. The gauge potential can be eliminated from the equations for a Higgs field in a similar way. If ϕ is a $\mathscr{G}$-valued function on Minkowski space–time satisfying the Higgs field equation

$$D_A^* D_A \phi + (\mu^2 + \lambda\,\mathrm{trace}(\phi(x)^*\,\phi(x))\,\phi(x) = 0 \qquad (5.30)$$

then the function $\Phi(\sigma) := P^A(\sigma)^{-1}\,\phi(\sigma(1))\,P^A(\sigma)$ satisfies

$$\partial_u\Phi(\sigma) + [B(\sigma, u), \Phi(\sigma)] = 0, \qquad \sigma \text{ in } \mathscr{P},\ u \text{ in } \mathscr{P}_0, \qquad (5.31)$$

and

$$D^i D_i \Phi(\sigma) + (\mu^2 + \lambda\,\mathrm{trace}(\Phi(\sigma)^*\,\Phi(\sigma)))\,\Phi(\sigma) = 0 \qquad (5.32)$$

as we have already seen in Eqs. (4.3), (4.4), and (4.5). Conversely, if L is a path 2-form satisfying (5.8) and (5.9) while Φ satisfies (5.31) and (5.32) then $L = L^A$ for some gauge potential A as in Theorem 5.1 and the function $\phi \equiv P^A(\sigma)\,\Phi(\sigma)\,P^A(\sigma)^{-1}$ depends only on the endpoint of σ and satisfies (5.30). The map $(A, \phi) \to (L^A, \Phi)$ is one to one modulo restricted gauge transformations.

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